Modeling beyond randomness

Andreas Stathopoulos

College of William and Mary
UMSA 2002

A stochastic problem

Let's go back to the 1-d drunkard's walk that we saw earlier this semester

Assume:

- the drunkard tends to go left with probability 45% and right with 55%
- there are 60 different discrete points on the 1-D road
- at the two ends of the road the drunkard still chooses a direction
  - if the direction is within the road the drunkard goes for it
  - if it is outside the road the drunkard stays at the same point

Question: After infinitely many steps what is the probability of the drunkard being at a specific point (state)?

A classical Markov chain problem

A stochastic simulation solution

Here is a simple (simplistic?) way of simulating the walk in matlab:

```
function drunk1d_rand(sample,states,time)

middle = ceil(states/2);
drunkards = middle*ones(sample,1);
State_plot = zeros(states,1);
State_plot(middle) = sample;

for t = 1:time
   for i = 1:sample
        coin = floor( rand(1,1) + 0.55 );
        State_plot(drunkards(i)) = State_plot(drunkards(i)) - 1;
        if (coin == 0)
          if (drunkards(i) > 1)
            drunkards(i) = drunkards(i) - 1;
          end
        else
          if (drunkards(i) < states)
            drunkards(i) = drunkards(i) + 1;
          end
        end
        State_plot(drunkards(i)) = State_plot(drunkards(i)) + 1;
   end
end;
plot( State_plot );
```

Evaluating stochastic simulation

- Easy to implement
- Relatively easy to enhance with additional complications of the model
- BUT Converges slowly !!

Question: How can we model this problem so we obtain analytic solutions?

A matrix formulation

Assume a set of N states $s_1, \ldots, s_N$.

Let the drunkard be at state s_j

Then the drunkard can choose to go to s_i with probability $p_{i,j}$, with $\sum_{i=1,N} p_{i,j} = 1$

Let's accumulate these column probabilities for each state in a matrix A.
A is called Probability matrix and note that:

$$[11 \cdots 1] * A = [11 \cdots 1]$$

.

For our drunkard example our matrix looks like this:

```
.45   .45     0
.55     0   .45  . . .
  0   .55     0
             . . . . .
             . . . . .
                        .55     0    .45
                              .55    .55
```

Turning it into an eigenvalue problem

Notice now that we look for the steady state probability, i.e., the probability that does not change from step to step (at infinity):

$$\text{Prob}(s_i) = \text{Prob}(s_{i-1})*.55 + \text{Prob}(s_{i+1})*.45$$

or equivalently, we want the eigenvector p with eigenvalue 1:

$$Ap = p$$

This is a 60×60 problem and can be solved in a jiffy numerically

However, note this is only for the steady state

Partial differential equations –more of the same

Consider the classical 1-D decceleration problem:

At time zero a rocket is fired from the ground upwards and after T seconds the rocket hits back the ground. What is the height of the rocket at any time in the interval $[0,T]$?

(For simplicity assume constant gravity acceleration = -10 m/sec^2 and $T = 1$)

This physical process can be described with a PDE involving the height $s(t)$:

$$\frac{d^2 s(t)}{dt^2} = -10$$

with the boundary conditions $s(0) = s(T) = 0$.

Analytic solution is trivial, but not for even slightly more complicated PDEs

The numerical approach –Discretization

Assume we want the solution only at a set of time points $t_i, i = 0, \ldots, N+1$

Assume equally spaced $t_i = i * h$ with $h = 1/(N+1)$ and $t_0 = 0, t_N = 1$

More accuracy can be obtained by increasing N

Since $s(t)$ continuous and doubly differentiable use Taylor series approximation

$$\frac{ds}{dt}(t) = \frac{s(t+h) - s(t)}{h} + \frac{h}{2}\frac{d^2 s(t)}{dt^2} + O(h^2)$$

Write the same based on the previous point approximation $t-h$ and substra

$$\frac{ds}{dt}(t) = -\frac{s(t-h) - s(t)}{h} - \frac{h}{2}\frac{d^2 s(t)}{dt^2} + O(h^2)$$

Substract the two equations:

$$\frac{d^2 s(t)}{dt^2} = \frac{s(t+h) - 2s(t) + s(t-h)}{h^2} + O(h^2)$$

The numerical approach –Discretization

We look for $s(t_i) \equiv s_i, i = 1, \ldots, N$. Then the formula becomes

$$\frac{d^2 s(t)}{dt^2} \approx \frac{1}{h^2}(s_{i-1} - 2s_i + s_{i+1})$$

and our equations become:

$$s_{i-1} - 2s_i + s_{i+1} = -h^2 10, \quad i = 1, \ldots, N$$

This is just a set of linear equations that can be represented in matrix form