Conclusions

Newton-Raphson Procedure

- Fails to converge for about $\frac{1}{3}$ of cases.
- Each iteration requires more computation than SIP.
- For cases in which it converges, Newton-Raphson normally requires more iterations.

Simple Iterative Procedure

- Always converges to a result for c^*.
- Almost always requires less iterations.
- Always requires less computation.

A Comparison

Table 1 contains a comparison of the two procedures.

An Illustration

The figure below illustrates the behavior of the SIP.

The Simple Iterative Procedure

- Finding c^* is now equivelent to finding a fixed point of $q(c)$.
- (McCool) $q(c)$ has only one fixed point, the unique solution c^*.

The Simple Iterative Procedure begins with

$$c_{k+1} = \frac{c_k + q(c_k)}{2} \tag{12}$$

starting at an initial point c_0 for $k = 0, 1, 2, \cdots$.

From Theorem 2 in the paper, we have the following:

- The SIP always converges for any choice $c_0 > 0$.
- The SIP converges with geometric rate $\frac{1}{2}$.

Creating the SIP

By using the new notation introduced on the previous slide, we can rewrite equation (2) as

$$\frac{s_3}{s_2} - \frac{1}{c} = \frac{s_1}{n}. \tag{9}$$

Solving the above equation for c, we obtain

$$c = \frac{ns_2}{ns_3 - s_1 s_2} \tag{10}$$

If we let

$$q(c) = \frac{ns_2}{ns_3 - s_1 s_2} \tag{11}$$

then we have reduced equation (2) to $q(c) = c$.

A New Look

To simplify our notations, let

$$s_1(c) = \sum_{i=1}^{n} \log x_i, \qquad s_2(c) = \sum_{i=1}^{n} x_i^c, \tag{6}$$

$$s_3(c) = \sum_{i=1}^{n} x_i^c \log x_i, \quad \text{and} \quad s_4(c) = \sum_{i=1}^{n} x_i^c \log^2 x_i. \tag{7}$$

So, we can write the Newton-Raphson procedure as

$$c_{k+1} = c_k - \frac{\frac{c}{n} s_1 s_2 - c s_3 + s_2}{\frac{1}{n} s_1 (s_2 + c s_3) - c s_4}. \tag{8}$$

Newton-Raphson Method used for Parameter Estimation

$$g(c) = \frac{c}{n} \sum_{i=1}^{n} \log x_i \sum_{i=1}^{n} x_i^c - c \sum_{i=1}^{n} x_i^c \log x_i + \sum_{i=1}^{n} x_i^c \tag{4}$$

$$g'(c) = \frac{1}{n} \sum_{i=1}^{n} \log x_i \left(\sum_{i=1}^{n} x_i^c + c \sum_{i=1}^{n} x_i^c \log x_i \right) - c \sum_{i=1}^{n} x_i^c \log^2 x_i \tag{5}$$

- (4) is arrived at by setting equation (2) equal to 0.
- (5) is simply the first derivitave of (4).

The Newton-Raphson Method

$$c_{k+1} = c_k - \frac{g(c_k)}{g'(c_k)}, \quad \text{for } k = 0, 1, 2, \cdots \tag{3}$$

- The value $|c_{k+1} - c_k|$ does not always converge to 0.
- Convergence depends upon the choice of the initial value, c_0.

Maximum Likelihood Estimators

$$b = \left[\frac{\sum_{i=1}^{n} x_i^c}{n} \right]^{\frac{1}{c}} \tag{1}$$

$$\frac{\sum_{i=1}^{n} x_i^c \log x_i}{\sum_{i=1}^{n} x_i^c} - \frac{1}{c} = \frac{1}{n} \sum_{i=1}^{n} \log x_i \tag{2}$$

- (McCool) A unique, positive solution c^* exists for (2).
- There is no known analytical method for solving (2).

Weibull Distribution

$$f(x; a, b, c) = \frac{c(x-a)^{c-1}}{b^c} \exp\left\{-\left(\frac{x-a}{b}\right)^c\right\}$$

where $x \geq a$, $b > 0$, $c > 0$.

The three parameters are

- a (location - set to 0 for this paper)
- b (scale)
- c (shape)

The Weibull pdf is useful as a failure model in analyzing the reliability of different types of systems.

Parameter estimation of the Weibull probability distribution

- Weibull Distribution
- Newton-Raphson Method
- Application to Weibull Parameter Estimation Problem
- Simple Iterative Procedure (SIP)
- Comparision of Both Procedures