The Distribution of the Longest Path in a Stochastic Activity Network: Simulation and Analytic Approaches

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Introduction: Activity networks

Notation

Nodes (vertices)	Events in time
n	Number of nodes
Arcs (edges)	Activities
m	Number of arcs
T_j	Time of completion of last activity entering node j
ij	Arc going from node i to node j
Y_{ij}	Duration of activity between nodes i and j

Sources

Elmaghraby, Salah E., *Activity Networks: Project Planning and Control by Network Models*, John Wiley & Sons, 1977.

Hagstrom, Jane N., Computing the Probability Distribution of Project Duration in a PERT Network, *Networks*, Vol. 20, 1990, pp 231-244.

Pritsker, A. Alan B., *Introduction to Simulation and SLAM II, Third Edition*, John Wiley & Sons, 1986.

Shier, Douglas R. *Network Reliability and Algebraic Structures*, Oxford University Press, 1991.

Simulation Approach:
Network representation as a matrix

Node-Arc Incidence Matrix: $N[n, m]$

$$\begin{bmatrix} 1 & 1 & 0 & 0 & 0 \\ -1 & 0 & 1 & 1 & 0 \\ 0 & -1 & -1 & 0 & 1 \\ 0 & 0 & 0 & -1 & -1 \end{bmatrix}$$

Further Notation

i is adjacent to j
j is adjacent from i

$\mathcal{B}(i)$	Set of nodes adjacent to i
$\mathcal{A}(i)$	Set of nodes adjacent from i
M	Set of paths from node 1 to node n
r	Number of paths in M
$\pi_1, \pi_2, ..., \pi_r$	Members of M
L_i	Path duration (excluding wait time)
π_c	Critical path
$p(\pi_k)$	Probability that $\pi_k \equiv \pi_c$
ρ_{ij}	Criticality of arc ij

Simulation Approach:
Point Estimators

Time to completion:

$$T_1 = 0.0$$

$$T_j = \max_{i \in \mathcal{B}(j)} (T_i + Y_{ij})$$

Critical path:

$$\pi_c \equiv \pi_k$$

where k is the index corresponding to the longest length path

$$\max(L_1, L_2, ..., L_r)$$

Criticality:

$$\rho_{ij} = \sum_{k=1}^{r} \left(p(\pi_k) \times \begin{cases} 1 & \text{if } ij \in \pi_k \\ 0 & \text{if } ij \notin \pi_k \end{cases} \right)$$

Simulation Approach:
Interval Estimators

Time to completion:

$$\bar{x} - \frac{t^* s}{\sqrt{n-1}} \le \mu \le \bar{x} + \frac{t^* s}{\sqrt{n-1}}$$

where

$$t^* = \texttt{idfStudent}(n-1, 1-\alpha/2).$$

Critical path and criticality:

$$\frac{1}{1 + \frac{n-y+1}{yF_{2y,2(n-y+1),1-\alpha/2}}} < p < \frac{1}{1 + \frac{n-y}{(y+1)F_{2(y+1),2(n-y),\alpha/2}}}$$

Simulation Approach: Algorithm
Global: Network N, Activity durations Y_{ij}
Procedure: T
Argument: node j

```
int i, k ← 1, l ← 0;
real t, t_max ← 0.0;
if(j = 1)
    return(0.0);
while(l < |B(i)|) {
    if(N[j, k] = -1) {
        i ← 1;
        while(N[i, k] ≠ 1)
            i++;
        t ← T_i + Y_ij;
        if(t ≥ t_max)
            t_max ← t;
        l++;
    }
    k++;
}
return(t_max);
```

Simulation Approach:
Results (Pritsker, pp 216)

Arc Index	Arc ij	$F_{Y_{ij}}(\tau)$
1	1,2	`Triangle`(1,5,3)
2	1,3	`Triangle`(3,9,6)
3	1,4	`Triangle`(12,19,13)
4	2,5	`Triangle`(3,12,9)
5	2,3	`Triangle`(1,8,3)
6	3,6	`Triangle`(8,16,9)
7	3,4	`Triangle`(4,13,7)
8	5,6	`Triangle`(3,9,6)
9	4,6	`Triangle`(1,8,3)

For 1,000,000 realizations:

Paths π_k	
k	Path
1	$1 \rightarrow 3 \rightarrow 6$
2	$1 \rightarrow 2 \rightarrow 3 \rightarrow 6$
3	$1 \rightarrow 2 \rightarrow 5 \rightarrow 6$
4	$1 \rightarrow 4 \rightarrow 6$
5	$1 \rightarrow 3 \rightarrow 4 \rightarrow 6$
6	$1 \rightarrow 2 \rightarrow 3 \rightarrow 4 \rightarrow 6$

Time to Completion T_j			
j	$\bar{x}$	s	h
1	0.0	–	–
2	3.114	0.666	0.001
3	7.498	1.339	0.003
4	16.123	1.923	0.004
5	11.651	1.394	0.003
6	20.859	2.023	0.004

Critical Path $p(\pi_k)$		
k	$\widehat{p}$	h
1	0.064	0.0005
2	0.165	0.0007
3	0.147	0.0007
4	0.185	0.0008
5	0.123	0.0006
6	0.316	0.0009

Criticality ρ_{ij}			
ij	Paths	$\widehat{p}$	h
1,2	2,3,6	0.628	0.0010
1,3	1,5	0.187	0.0008
1,4	4,	0.185	0.0008
2,5	3	0.147	0.0007
2,3	2,6	0.481	0.0010
3,6	1,2	0.229	0.0008
3,4	5,6	0.439	0.0010
5,6	3	0.147	0.0007
4,6	4,5,6	0.624	0.0010

Analytical Methods:
Finding the Distribution of the Critical Path in a Series-Parallel Network

Consider two arcs in *parallel* between nodes i and j. Let X_{ij} denote the duration of one arc, and Y_{ij} the duration of the other. Without loss of generality, if $T_i = 0.0$ then $T_j = \max(X_{ij}, Y_{ij})$. So

$$\begin{aligned}
F_{T_j}(\tau) &= \Pr[T_j \le \tau] \\
&= \Pr[\max(X_{ij}, Y_{ij}) \le \tau] \\
&= \Pr[X_{ij} \le \tau \text{ and } Y_{ij} \le \tau] \\
&= \Pr[X_{ij} \le \tau] \times \Pr[Y_{ij} \le \tau] \\
&= F_{X_{ij}}(\tau) F_{Y_{ij}}(\tau).
\end{aligned}$$

Analytical Methods:
Finding the Distribution of the Critical Path in a Series-Parallel Network

Consider two arcs in *series* between nodes i, j, and k. Let Y_{ij} and Y_{jk} denote the durations of the two arcs with CDFs $F_{Y_{ij}}(\tau)$ and $F_{Y_{jk}}(\tau)$. Without loss of generality, if $T_i = 0.0$ then $T_k = Y_{ij} + Y_{jk}$. The CDF of T_k is

$$
\begin{aligned}
F_{T_k}(\tau) &= \Pr[T_k \leq \tau] \\
&= \int_0^{\tau} F_{Y_{ij}}(\tau - y) \partial F_{Y_{jk}}(y)
\end{aligned}
$$

Analytical Methods: Algorithm

Procedure: Decompose
Arguments: Network $N[n,m]$,
Activity durations $Y[m]$

int i, j, k, p, q, s_r, s_a;
boolean c;
$c \leftarrow$ true;
while($c =$ true) {
    $c \leftarrow$ false;
    for(i from 1 to n) {
        $s_r \leftarrow 0$;
        $s_a \leftarrow 0$;
        for(j from 1 to m) {
            $s_r \leftarrow s_r + N[i,j]$;
            $s_a \leftarrow s_a + |N[i,j]|$;
            if($N[i,j] =$ -1) $q \leftarrow j$;
            if($N[i,j] = 1$) $p \leftarrow j$;
        }
        if($s_r = 0$ and $s_a = 2$) {

```
                Y[q] ← Convolution(Y[q], Y[p]);
                for(k for 1 to n) {
                    N[k, q] ← N[k, q] + N[k, p];
                    N[k, p] ← 0;
                }
                c ← true;
            }
        }
        for(j from 1 to m) {
            for(k from j + 1 to m) {
                if(col(j, N) = col(k, N)) {
                    Y_j ← max(Y_j, Y_k);
                    col(k, N) = 0;
                    c ← true;
                }
            }
        }
    }
return(N, Y);
```

Analytical Methods:
Results

If the duration of each arc Y_{ij} is an $\texttt{Exponential}(b)$ random variable then the CDF of T_5 is

$$
\begin{aligned}
&F_{T_5}(\tau) = \\
&\quad -3tbe^{bt} - 1/2b^2e^{bt}t^2 - 3e^{-2bt} + \\
&\quad 5/2t^2b^2e^{-2bt} + 1/2t^3b^3e^{-2bt} + 2e^{-3bt} + \\
&\quad 3tbe^{-3bt} + t^2b^2e^{-3bt} + 1 \qquad t > 0
\end{aligned}
$$