CS 420-02: Undergraduate Simulation, Modeling and Analysis

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Chapter 1

Microeconomic Algorithms

CS 420-02: Undergraduate Simulation, Modeling and Analysis
1.1 Introduction

• In earlier lectures, we have seen how natural phenomena serve as metaphors for algorithm design:
  – Simulated Annealing: metaphor from metallurgy.

This lecture: metaphor from economics.
Application: fair division of (divisible) resources in a distributed system using a pricing mechanism.

• Key ideas: use a market-based approach

Consider this example:
  – A database file is shared by 3 branches of a bank.
  – Each branch would like to keep the entire file
    ⇒ better for local access.
  – Resource allocation problem: assuming file is divisible, how to decide a fair allocation?

Using a market mechanism:
  – Provide each branch with some play-money.
  – Each branch bids for as large a piece of the file as its “budget” allows.
  – Prices are adjusted iteratively, until each branch buys all it can, and whole file is spread across branches.
Why use a market approach?

- Decision-making is decentralized (good fault-tolerance).
- Market approach can accomodate multiple “commodities.”
- Algorithm is very simple (as we’ll see).

- **Outline of lecture:**

  - *Optimization 101*: the Lagrange Multiplier Method.
  - *Econ 101*: basics of microeconomics:
    * Producers.
    * Consumers.
    * A simple exchange economy.
    * Tattonnement.
  - Tattonnement as an algorithm.
  - An example.
1.2 Optimization 101: The Lagrange Multiplier Method

- **Focus**: optimization of a multivariable function subject to an equality constraint:

  \[
  \begin{align*}
  \text{maximize} & \quad U(x_1, x_2, \ldots, x_n) \\
  \text{s.t.} & \quad g(x_1, x_2, \ldots, x_n) = 0.
  \end{align*}
  \]

  In the optimization field, \( U \) is usually called the *objective function*. In microeconomics: \( U \) is often called the *utility function*.

- **Single-variable, unconstrained**: 
  - Method: set derivative to zero.

- **Multiple variables, unconstrained**: 
  - Method: set partial derivatives to zero.

- **Multiple variables, equality constraints**: 
  - Consider a 2-variable problem:

    \[
    \begin{align*}
    \text{maximize} & \quad U(x_1, x_2) \\
    \text{s.t.} & \quad g(x_1, x_2) = 0
    \end{align*}
    \]

    - Example of a problem:

      \[
      \begin{align*}
      \text{maximize} & \quad U(x_1, x_2) = 3x_1 + 4x_2 + x_1x_2 \\
      \text{s.t.} & \quad 6x_1 + 2x_2 - 30 = 0
      \end{align*}
      \]
- Consider now a small change in \( x = (x_1, x_2) \):

\[
x' = (x_1 + \Delta x_1, x_2 + \Delta x_2)
\]

such that the constrained is maintained:

\[
g(x_1 + \Delta x_1, x_2 + \Delta x_2) = 0.
\]

Then (first-order approximation):

\[
\frac{\partial g}{\partial x_1} \Delta x_1 + \frac{\partial g}{\partial x_2} \Delta x_2 = 0
\]

This change causes a change in \( U \):

\[
\Delta U = \frac{\partial U}{\partial x_1} \Delta x_1 + \frac{\partial U}{\partial x_2} \Delta x_2
\]

Combining, we get

\[
\Delta U = \Delta x_2 \left( \frac{\partial U}{\partial x_2} - \frac{\partial U}{\partial x_1} \frac{\partial g}{\partial x_2} \frac{\partial g}{\partial x_1} \right)
\]

\[
\Delta U \triangleq \Delta x_2 D
\]

Now, if \( D > 0 \), pick \( \Delta x_2 > 0 \) or if \( D < 0 \), pick \( \Delta x_2 < 0 \).

\( \Rightarrow \) at maximum \( D = 0 \).

Thus,

\[
\frac{\partial U}{\partial x_1} / \frac{\partial g}{\partial x_2} = \frac{\partial g}{\partial x_1} / \frac{\partial g}{\partial x_2}
\]

or, equivalently,

\[
\frac{\partial U}{\partial x_1} / \frac{\partial g}{\partial x_2} = \frac{\partial U}{\partial x_1} / \frac{\partial g}{\partial x_2} = \text{some constant } \lambda
\]
– Example:

\[
\text{maximize} \quad U(x_1, x_2) \\
\text{s.t.} \quad p_1 x_1 + p_2 x_2 = b
\]

Then, at the maximum

\[
\frac{\partial U}{\partial x_1} = \frac{p_1}{p_2}
\]

Interpretation: ratio of marginal utilities equals the price ratio at the optimum.

• The Lagrange Multiplier Method:

– Used to solve a problem with equality constraints, e.g.

\[
\text{maximize} \quad U(x_1, x_2) \\
\text{s.t.} \quad g(x_1, x_2) = 0
\]

– Method: define a new function

\[
L = U(x_1, x_2) - \lambda g(x_1, x_2)
\]

and find the unconstrained optimum of this function:

\[
\text{Set } \frac{\partial L}{\partial x_1} = 0, \quad \frac{\partial L}{\partial x_2} = 0, \quad \frac{\partial L}{\partial \lambda} = 0
\]

– Why does this work? Set the derivatives to zero as indicated above and see for yourself.
Example:

\[
\begin{align*}
\text{maximize } & \quad U(x_1, x_2) = 3x_1 + 4x_2 + x_1x_2 \\ 
\text{s.t. } & \quad 6x_1 + 2x_2 - 30 = 0
\end{align*}
\]

The Lagrangean function for this problem is:

\[
L = 3x_1 + 4x_2 + x_1x_2 - \lambda(6x_1 + 2x_2 - 30)
\]

To solve, set each derivative to zero:

\begin{align*}
\frac{\partial L}{\partial x_1} &= 0 \quad \Rightarrow \quad 3 + x_2 - 6\lambda = 0 \\
\frac{\partial L}{\partial x_2} &= 0 \quad \Rightarrow \quad 4 + x_1 - 2\lambda = 0 \\
\frac{\partial L}{\partial \lambda} &= 0 \quad \Rightarrow \quad 6x_1 + 2x_2 - 30 = 0
\end{align*}

Solution (3 equations, 3 unknowns): \(x_1 = 1, x_2 = 12\).

Note: it is common to write the optimal solution as \(x^* = (x_1^*, x_2^*) = (1, 12)\).
1.3 \textbf{ECON 101: Introduction}

- \textbf{What is Economics?}  
  Formal study of resource allocation, individual vs. social objectives, mechanisms and institutions, models and policy.

- \textbf{Areas of economics:}
  - \textit{Microeconomics}: (detailed picture of individuals)  
    * Theory of economics at the level of individual agents, e.g. individual consumers, firms, investors.  
    * General theory assumes rational behavior (selfish optimization) and perfect information.
  - \textit{Macroeconomics}: (broad picture)  
    * Theory (and practice) of aggregates, e.g., inflation, unemployment, GDP.
  - Other specialized areas:  
    * Institutions: central bank, stock exchange.  
    * Government policy: taxation, social and infrastructure spending, monetary policy.  
    * International economics: trade, protection, tariffs.  
    * Development economics: economic paradigms for developing nations.
• **Key elements of microeconomics:**
  
  – Divisible commodities:
    * Assume a commodity is divisible (real-valued).
    * Let \( \mathbf{x} = (x_1, \ldots, x_k) \) represent an *allocation* of \( k \) commodities:
      \[
      x_i = \text{amount of commodity } i
      \]
      \( \mathbf{x} \) is called a *commodity bundle*.
  
  – Analysis of Producers:
    * Cost of production of a commodity.
    * Profit motive and profit maximization.
  
  – Analysis of Consumers:
    * Preferences amongst commodities.
    * Utility functions and utility maximization.
  
  – Market economy:
    * Exchange economies and prices.
    * **Def:** Market. A collection of consumers and/or producers and a mechanism for trade.
    * **Def:** Perfect market. Agents (consumers, producers) are rational, have perfect information and engage in unrestricted trade. Also, no agent is powerful enough to influence prices in isolation.
1.4 ECON 101: The Producer

- Think of a producer as a firm manufacturing one or more commodities.
  **Goal:** a simple mathematical characterization of a producer.

- **Modeling a producer using a production function:**
  - The producer is a function from input commodities to output commodities:
    
    $z_1 \rightarrow y_1$
    $z_2 \rightarrow y_2$
    ...
    $z_m \rightarrow y_n$

    - $z_i$ = amount of input commodity $i$
    - $y_j$ = amount of output commodity $j$

  - Typical assumption (for simplicity): only one output
    
    $z_1 \rightarrow y = f_p(z_1, \ldots, z_m)$

  - Another assumption: analysis is static (for a fixed period of time).
- Example:

\[ y = f_p(z_1, z_2) = \gamma \sqrt{z_1} \sqrt{z_2} \]

where

\[ y = \text{wheat produced} \]
\[ z_1 = \text{labor} \]
\[ z_2 = \text{fertilizer/pesticide} \]

- The cost function of a producer:

  - Suppose a producer uses inputs \( z = (z_1, \ldots, z_m) \).
  
  - If the unit price for commodity \( z_i \) is \( p_i \).
  
    \( \Rightarrow \) producer’s cost is \( p_1 z_1 + \ldots + p_m z_m = \mathbf{p} \cdot \mathbf{z} \).

  - Suppose the desired level of output is \( y \).

    Then, to minimize costs, the producer solves the following optimization problem:

    \[
    \begin{align*}
    \text{minimize} & \quad p_1 z_1 + \ldots + p_m z_m \\
    \text{s.t.} & \quad f_p(z_1, \ldots, z_m) = y
    \end{align*}
    \]

- Example: \( f_P(z_1, z_2) = \sqrt{z_1} \sqrt{z_2} \), and the desired output level is \( k \) units

  \( \Rightarrow \) the following problem is solved

    \[
    \begin{align*}
    \text{minimize} & \quad p_1 z_1 + p_2 z_2 \\
    \text{s.t.} & \quad \sqrt{z_1} \sqrt{z_2} = k
    \end{align*}
    \]

Solve this to get \( z^* = (z_1^*, z_2^*) = (k \sqrt{\frac{z_1}{p_1}}, k \sqrt{\frac{z_2}{p_2}}) \).
– The *cost function* of the producer is the cost of the optimal solution to the above problem:

\[ C(p, y) = \min_z p \cdot z \quad s.t. \quad f_p(z) = y \]

In the above example,

\[ C(p_1, p_2, k) = p_1 k \sqrt{\frac{p_2}{p_1} + p_2 k} \sqrt{\frac{p_1}{p_2}} \]

– Properties of the cost function:
  * Linearity in prices: \( C(\alpha p, y) = \alpha C(p, y) \).
  * \( C(p, y) \) is non-decreasing in \( p \).
  * \( C(p, y) \) is concave in \( p \).

• Producers in the presence of consumers:
  – So far, producers have been studied in isolation.
    Now we consider demand and profit maximization.
  – The *demand function*:
    * Generally, if a product’s price goes up, the demand for that product goes down.
    * Let \( D(p) \) = demand seen by a producer at price \( p \).
    * \( D \) is called the demand function.
  – The *inverse demand function*:
    * Assume \( D \) is invertible: \( p = D^{-1}(d) \) is called the inverse-demand function, written as \( p = P(d) \), for a particular demand value \( d \). It answers the question: what do I set the price to be to create the demand \( d \)?
  – Maximizing profit:
    * Assume \( y \) units are made and all units are sold
      \( \Rightarrow \) revenue \( R(y) = y \times \text{price per unit} \).
     But, at production level \( y \), the price should be set at
     \[ P(y) = D^{-1}(y) \]
At this price, revenue is

\[ R(y) = yP(y) \]

* If \( C(y) \) is the cost of producing \( y \) units (from cost function)

\[ \text{profit } \pi(y) = R(y) - C(y) = yP(y) - C(y) \]

– The basic law of production:
  * To maximize profit, set derivative of \( \pi(y) \) to zero and obtain

\[ \frac{dR(y)}{dy} = \frac{dC(y)}{dy}. \]

* Basic law of production: at maximum profit, marginal revenue equals marginal cost.
* Interpretation: if additional revenue obtained from a slight increase in production exceeds cost increase
  \[ \Rightarrow \] you would produce more.

• Producers in competition:
  – Currently, we think of a producer maximizing profit by determining output level \( y \) to maximize profit \( \pi(y) \).
    \[ \Rightarrow \] assumes a firm can determine prices via \( P(y) \) function
  – In perfect competition, a firm cannot determine prices
    \[ \Rightarrow \] firms are price-takers
    (Any firm that increases prices will not be able to sell anything).
  – If prices are given,

\[ \pi(y) = yp - C(y). \]

– Let \( Y(p) \) the output level that maximizes \( \pi(y) \) for given price \( p \).
– \( Y(p) \) is called the supply function.
– Usually, in competition, each firm \( i \) has a supply function \( Y_i(p) \).
  Then, \( Y(p) = \Sigma_i Y_i(p) \) is called the aggregate supply function.
- Generally, $Y(p)$ increases with $p$.
- Generally, $D(p)$ decreases with $p$ (Demand decreases with increasing price).
- Equilibrium is obtained when $D(p) = Y(p)$ (Supply = demand).
1.5 ECON 101: The Consumer

- The consumer in isolation: modeling via preferences
  - How do we model consumer behavior?
    One approach: define a preference relation.
  - Notation: suppose \( \mathbf{x} = (x_1, \ldots, x_k) \) and \( \mathbf{y} = (y_1, \ldots, y_k) \) are commodity bundles.
    Example:
    \[
    (100,20) \Rightarrow 100 \text{ units of beer, 20 units of chips} \\
    (40,30) \Rightarrow 40 \text{ units of beer, 30 units of chips}
    \]
    Which is a better combination?
  - Each consumer has a preference relation that allows commodity bundles to be compared:
    \( \mathbf{x} \leq \mathbf{y} \Rightarrow \mathbf{y} \) is at least as good as \( \mathbf{x} \).
  - Define the \textit{indifference} relation \( \mathbf{x} \sim \mathbf{y} \) if both \( \mathbf{x} \leq \mathbf{y} \) and \( \mathbf{y} \leq \mathbf{x} \).
    Define \( C_\mathbf{x} \triangleq \{ \mathbf{y} : \mathbf{x} \sim \mathbf{y} \} \).
    \( C_\mathbf{x} \) is called an \textit{indifference curve} – the locus of points with equivalent preference.
  - Desirable properties of preference relations:
    * \textit{Continuity}: continuous indifference curves.
    * \textit{Monotonicity}: if \( \mathbf{x} \succeq \mathbf{y} \), then \( \mathbf{x} \succeq \mathbf{y} \).
      (More is better).
* **Convexity:** if $\mathbf{x} \preceq \mathbf{z}$ and $\mathbf{y} \preceq \mathbf{z}$ then $\alpha \mathbf{x} + (1 - \alpha) \mathbf{y} \preceq \mathbf{z}$. 
(0 $\leq \alpha \leq 1$).

(Indifference curves are convex)

![Diagram of convex indifference curves]

Interpretation: a balanced allocation is better than an extreme one.

- **The consumer in isolation: modeling via utility functions**
  - **Motivation:** it’s easier to work with functions than relations.
  - **Def:** A utility function is a function $U$ s.t.
    $$U(\mathbf{x}) \geq U(\mathbf{y}) \Leftrightarrow \mathbf{x} \succeq \mathbf{y}.$$  
  - Desirable properties of utility functions:
    * $U$ should be continuous.
    * $U$ should be monotonic:
      $$\mathbf{x} \succeq \mathbf{y} \Leftrightarrow U(\mathbf{x}) \geq U(\mathbf{y}).$$  
      (More is better).
    * $U$ should be quasi-concave:  
      $$U(\mathbf{x}) \geq c \text{ and } U(\mathbf{y}) \geq c \Leftrightarrow U(\alpha \mathbf{x} + (1 - \alpha) \mathbf{y}) \geq c.$$  
  - **Fact:** if the preference relation is continuous and monotonic, then a continuous utility function exists.
  - **Fact:** $U$ quasi-concave $\Leftrightarrow$ convex indifference curves.

- **The consumer and prices:**
  - Suppose commodity $i$ has unit price $p_i$.  

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- Assume consumer has budget $b$.
- The consumer’s behavior is modeled as:

$$
\text{maximize } \quad U(x) \\
\text{s.t. } \quad \sum_i p_i x_i \leq b.
$$

- Example:

$$
U(x_1, x_2) = 8x_1 + 4x_2 + 2x_1x_2 \\
p_1 = 5 \\
p_2 = 10 \\
b = 100
$$

Hence the consumer solves

$$
\text{maximize } 8x_1 + 4x_2 + 2x_1x_2 \\
\text{s.t. } 5x_1 + 10x_2 = 100
$$

Solution turns out to be: $x_1^* = 13.25, x_2^* = 4.375$.

- The consumer’s demand function:
  - Consider a consumer with utility function $U$ and budget $b$.
  - We assume the consumer solves

$$
\text{maximize } \quad U(x) \\
\text{s.t. } \quad \sum_i p_i x_i \leq b
$$

The solution depends on both $p$ and $b$.
- Let $X_U(p, b)$ be the solution you get for prices $p$ and budget $b$.
- $X_U(p, b)$ is called the demand function.
- Generally, $X_U(p, b)$ decreases with increasing $p$. 
1.6 ECON 101: An Exchange Economy

- **What is an exchange economy?**
  - Consider a market with many consumers and NO producers.
  - Assume $k$ commodities.
  - Consumer $i$ has initial endowment $\mathbf{x}^i = (x_1^i, \ldots, x_k^i)$, i.e., $x_l^i$ amount of commodity $l$.
  - What kind of economic activity is possible without producers?
    - consumers can trade
    - called an *exchange economy*

- **Example:** consider two consumers (A and B) and two commodities
  - Suppose consumer A has utility function $U_A(x_A^1, x_A^2) = 4x_A^1 + 2x_A^2$ and initial endowment (4,12).
    - initial utility of A $= U_A(4, 12) = 40$.
  - Suppose consumer B has utility function $U_B(x_B^1, x_B^2) = x_B^1 + 2x_B^2$ and initial endowment (16,8).
    - initial utility of B $= U_B(16, 8) = 32$.
  - Note: $x_A^1 + x_B^1 = 20$ and $x_A^2 + x_B^2 = 20$.
  - Suppose A and B consider trading so that
    * A gets (12,4) and
    * B gets (8,16).
    What are the new utilities? Should they trade?
  - Suppose A and B consider trading so that
    * A gets (6,8) and
    * B gets (14,12).
What are the new utilities? Should they trade?

- Assumption: a trade will occur if at least one improves without hurting the other.

- What about when
  * A gets (8,16) and
  * B gets (12,4)?

- Suppose the initial endowments were such that
  * A has (20,20) and
  * B has (0,0).

  There is no trade that A could possibly agree to.

- **Def**: An allocation is *Pareto-optimal* if there is no trade that improves the utility of at least one guy while not hurting the others.

- **NOTE**:

  - Pareto-optimality is a way of evaluating the value of an allocation.
  
  - An alternative way is to define a society-wide utility function in terms of individual utility functions.
    
    e.g. \( U_S(x^A, x^B) = U_A(x^A) + U_B(x^B) \).
  
  - In this case, the “best” allocation is the one that minimizes \( U_S \).
  
  - However, the sum-of-utilities is somewhat artificial
    
    \( \Rightarrow \) product-of-utilities will give a different result.
  
  - Pareto-optimality instead uses only individual utilities and considers allocations from a selfish-individual perspective.
• 2 consumers and 2 commodities: some analysis

- Suppose
  * Consumer A has initial endowment \((\bar{x}_1^A, \bar{x}_2^A)\).
  * Consumer B has initial endowment \((\bar{x}_1^B, \bar{x}_2^B)\).
  
  with totals \(S_1 = \bar{x}_1^A + \bar{x}_1^B\) and \(S_2 = \bar{x}_2^A + \bar{x}_2^B\).

- To achieve Pareto-optimality, consumer A solves

  \[
  \begin{align*}
  \text{maximize} & \quad U_A(x_1^A, x_2^A) \\
  \text{s.t.} & \quad x_1^A + x_1^B = S_1 \\
  & \quad x_2^A + x_2^B = S_2 \\
  & \quad U_B(x_1^B, x_2^B) \geq U_B(\bar{x}_1^B, \bar{x}_2^B)
  \end{align*}
  \]

  (Consumer B solves a similar problem).

  The Lagrange Multiplier Method is used to obtain:

  \[
  \frac{\partial U_A/\partial x_1^A}{\partial U_A/\partial x_2^A} = \frac{\partial U_B/\partial x_1^B}{\partial U_B/\partial x_2^B}
  \]

  - Interpretation: at the optimum, marginal rates of substitution are the same across all individuals.

  If they are not, a small trade would improve both individuals’ utilities.
Consider an exchange economy with 2 commodities and 2 consumers:

- Suppose the individuals cannot affect prices in isolation (say, an external referee selects prices).
- Let the prices be $p = (p_1, p_2)$ for the two goods.
- Suppose
  * Consumer A has initial endowment $(\bar{x}_1^A, \bar{x}_2^A)$.
  * Consumer B has initial endowment $(\bar{x}_1^B, \bar{x}_2^B)$.

Then,

- A’s initial budget (worth) is $b_A = p_1\bar{x}_1^A + p_2\bar{x}_2^A$.
- B’s initial budget (worth) is $b_B = p_1\bar{x}_1^B + p_2\bar{x}_2^B$.

- **Def**: The price-allocation combination of

\[
\begin{align*}
\mathbf{x}^A &= (x_1^A, x_2^A) \\
\mathbf{x}^B &= (x_1^B, x_2^B) \\
p &= (p_1, p_2)
\end{align*}
\]

is called a *competitive equilibrium* if

1. the allocations are feasible, i.e.,

\[
\begin{align*}
x_1^A + x_1^B &= \bar{x}_1^A + \bar{x}_1^B \\
x_2^A + x_2^B &= \bar{x}_2^A + \bar{x}_2^B
\end{align*}
\]
2. the allocations are budget-feasible, i.e.,

\[
p_1 x_1^A + p_2 x_2^A = b_A \\
p_1 x_1^B + p_2 x_2^B = b_B
\]

3. for every other budget-feasible allocation \( y^A, y^B \), the following is true:

\[
U_A(y^A) \leq U_A(x^A) \\
U_B(y^B) \leq U_B(x^B)
\]

- **First Fundamental Theorem of (Welfare) Economics:**
  
  A price-allocation combination \((x^A, x^B, p)\) that satisfies competitive equilibrium is Pareto-optimal.

  (Technical assumptions need to be made, e.g., quasi-concave utility functions).

- **Second Fundamental Theorem of (Welfare) Economics:**
  
  If the allocation \( x^A, x^B \) is Pareto-optimal, then there is a price vector \( p \) such that \((x^A, x^B, p)\) is a competitive equilibrium.

  (Similar technical assumptions).

- **NOTE:**
  
  - Pareto-optimality is not defined in terms of prices.
    
    The theorems ensure that linear (per-unit) pricing allows one to achieve Pareto-optimality through price-constrained selfish optimization.
  
  - The result holds for multiple commodities and consumers.
  
  - A more general result includes the presence of producers.
1.8 Tattonnement

- So far, we only only discussed the existence of equilibrium prices.
  
  **Key question:** how to implement a mechanism to find the equilibrium prices (and hence, a Pareto-optimal allocation)?

- **Omniscient dictatorship approach:**
  - A dictator uses the necessary conditions for Pareto-optimality and computes the equilibrium prices and allocations.
  - Consumers are informed of their optimal allocations.
  - Consumers then make exchanges to achieve the optimal allocation.

  Drawback of this method: presumes the existence of an omniscient dictatorship.

- **Tattonnement:** a decentralized implementation of price-determination.

- **Key ideas in Tattonnement:**
  - The referee selects arbitrary an initial price for each commodity.
  - Each consumer maximizes his/her utility within budget constraints to obtain his/her desired allocation.
  - Each consumer reports his/her desired allocation to a referee.
  - The referee looks at the total amount requested for each commodity:
    * If the total amount requested is *more* than the total available, the referee increases the price for that commodity.
    * If the total amount requested is *less* than the total available, the referee decreases the price for that commodity.
    * Otherwise, price is unchanged.
  - If any one price changed, the referee reports the new prices to the consumers.
– The process is repeated until prices have converged.
– The limiting prices are taken as the equilibrium prices.
– Finally, the allocations are determined based on these equilibrium prices.

NOTE:

– Utility computations are decentralized (each consumer computes his/her own utility maximization).
– High demand for a commodity increases its price.
– Low demand decreases the price.
1.9 Tattonnement as an Algorithm

- **Observation**: Tattonnement can be used as a distributed algorithm for resource allocation in a distributed system.

- **Example**:
  - Suppose 2 branches of a bank wish to share 2 database files.
    - \( \Rightarrow \) 2 commodities (the 2 files) and 2 consumers (the 2 branches).
  - Suppose the files are accessed by queries generated locally.
    - * Some queries generated at A only access File 1, others access only File 2.
    - * Some queries generated at B access both files.
    - * The same holds for queries generated at B.
  - Suppose the following probabilities are known (via estimation, say)
    
    \[
    \begin{align*}
    \alpha_1^A &= P [\text{An access at A is only for File 1}] \\
    \alpha_2^A &= P [\text{An access at A is only for File 2}] \\
    \alpha_{12}^A &= P [\text{An access at A is for both files}]
    \end{align*}
    \]

    Here, \( \alpha_1^A + \alpha_2^A + \alpha_{12}^A = 1 \).
  - Similarly, define
    
    \[
    \begin{align*}
    \alpha_1^B &= P [\text{An access at B is only for File 1}] \\
    \alpha_2^B &= P [\text{An access at B is only for File 2}] \\
    \alpha_{12}^B &= P [\text{An access at B is for both files}]
    \end{align*}
    \]

  - Next, assume the files are divisible:
* $x_1^A$ = fraction of File 1 stored at A.
* $x_2^A$ = fraction of File 2 stored at A.

- Each consumer would like to maximize the probability that an access is locally satisfied.

\[ P \text{ [An access at A is locally satisfied]} = \alpha_1^A x_1^A + \alpha_2^A x_2^A + \alpha_1^B x_1^B x_2^A \]

(Similar expression for B).

- Thus, the utility functions of the consumers are:

\[
U_A(x_1^A, x_2^A) = \alpha_1^A x_1^A + \alpha_2^A x_2^A + \alpha_1^B x_1^A x_2^A
\]

\[
U_B(x_1^B, x_2^B) = \alpha_1^B x_1^B + \alpha_2^B x_2^B + \alpha_1^B x_1^B x_2^B
\]

- Thus, given prices $(p_1, p_2)$, consumer A solves

\[
\text{maximize } \alpha_1^A x_1^A + \alpha_2^A x_2^A + \alpha_1^A x_1^A x_2^A \\
\text{s.t. } p_1 x_1^A + p_2 x_2^A \leq b_A \\
0 \leq x_1^A \leq 1, \quad 0 \leq x_2^A \leq 1.
\]

Using the Lagrange Multiplier Method, consumer A obtains

\[
x_1^A(p_1, p_2) = \frac{\alpha_1^A b_A + \alpha_1^A p_2 - \alpha_2^A p_1}{2\alpha_1^A p_1}
\]

\[
x_2^A(p_1, p_2) = \frac{\alpha_1^A b_A + \alpha_2^A p_1 - \alpha_1^A p_2}{2\alpha_1^A p_2}
\]

(Consumer B solves a similar problem)
The basic algorithm:

<table>
<thead>
<tr>
<th>Algorithm: Tatonnement()</th>
</tr>
</thead>
<tbody>
<tr>
<td>1. for i := 1 to num_commodities</td>
</tr>
<tr>
<td>2. p[i] := initial price of i-th commodity;</td>
</tr>
<tr>
<td>3. total_amount[i] := total amount of i-th commodity;</td>
</tr>
<tr>
<td>4. endfor</td>
</tr>
<tr>
<td>5. for n := 1 to num_iterations</td>
</tr>
<tr>
<td>// Use current prices p to maximize utility</td>
</tr>
<tr>
<td>6. Compute amounts x^A[i] that maximize A's utility;</td>
</tr>
<tr>
<td>7. Compute amounts x^B[i] that maximize B's utility;</td>
</tr>
<tr>
<td>8. for i := 1 to num_commodities</td>
</tr>
<tr>
<td>9. total_demand[i] := x^A[i] + x^B[i];</td>
</tr>
<tr>
<td>10. p[i] := p[i] + \eta (total_amount[i] - total_demand[i]);</td>
</tr>
<tr>
<td>11. endfor</td>
</tr>
<tr>
<td>12. endfor</td>
</tr>
</tbody>
</table>
Another example: how to finesse the handling of indivisible commodities

- Suppose now that each of A and B above also want to share a printer.
- How is a printer to be divided?
- One way of “sharing” a printer:
  * Whenever A sends a print job to the queue, a coin with $P[\text{heads}] = y^A$ is flipped.
    If heads is obtained, all of A’s jobs are moved to the head of the queue.
    If tails is obtained, A’s new job joins the end of the print queue.
  * A similar $y^B$-biased coin is associated with B.
  * We will enforce $y^A + y^B = 1$
    (even though it’s enough to ensure $y^A + y^B = \text{constant}$).
- Then, along with the two files, A’s utility function is

$$U_A(x_1^A, x_2^A) = \alpha_1^A x_1^A + \alpha_2^A x_2^A + \alpha_{12}^A x_1^A x_2^A + \beta^A y^A$$

where $\beta^A$ is a constant.
- Similarly, B’s utility function is

$$U_B(x_1^B, x_2^B) = \alpha_1^B x_1^B + \alpha_2^B x_2^B + \alpha_{12}^B x_1^B x_2^B + \beta^B y^B$$