CS 420-02: Undergraduate Simulation, Modeling and Analysis

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Chapter 1

Microeconomic Algorithms

CS 420-02: Undergraduate Simulation, Modeling and Analysis

1.1 Introduction

- In earlier lectures, we have seen how natural phenomena serve as metaphors for algorithm design:
  - Genetic Algorithms: metaphor from biology.
  - Simulated Annealing: metaphor from metallurgy.

  This lecture: metaphor from economics.

  Application: fair division of (divisible) resources in a distributed system using a pricing mechanism.

- **Key ideas**: use a market-based approach

  Consider this example:
  - A database file is shared by 3 branches of a bank.
  - Each branch would like to keep the entire file
    $\Rightarrow$ better for local access.
  - Resource allocation problem: assuming file is divisible, how to decide a fair allocation?

  Using a market mechanism:
  - Provide each branch with some play-money.
  - Each branch bids for as large a piece of the file as its "budget" allows.
  - Prices are adjusted iteratively, until each branch buys all it can, and whole file is spread across branches.

Why use a market approach?

- Decision-making is decentralized (good fault-tolerance).
- Market approach can accomodate multiple "commodities."
- Algorithm is very simple (as we'll see).

- **Outline of lecture**:
  - *Optimization 101*: the Lagrange Multiplier Method.
  - *Econ 101*: basics of microeconomics:
    * Producers.
    * Consumers.
    * A simple exchange economy.
    * Tatonnement.
  - Tatonnement as an algorithm.
  - An example.

1.2 Optimization 101: The Lagrange Multiplier Method

- **Focus**: optimization of a multivariable function subject to an equality constraint:

$$\begin{aligned} \text{maximize} \quad & U(x_1, x_2, \ldots, x_n) \\ s.t. \quad & g(x_1, x_2, \ldots, x_n) = 0. \end{aligned}$$

  In the optimization field, U is usually called the *objective function.*

  In microeconomics: U is often called the *utility function.*

- **Single-variable, unconstrained**:
  - Method: set derivative to zero.
- **Multiple variables, unconstrained**:
  - Method: set partial derivatives to zero.
- **Multiple variables, equality constraints**:
  - Consider a 2-variable problem:

$$\begin{aligned} \text{maximize} \quad & U(x_1, x_2) \\ s.t. \quad & g(x_1, x_2) = 0 \end{aligned}$$

  - Example of a problem:

$$\begin{aligned} \text{maximize} \quad & U(x_1, x_2) = 3x_1 + 4x_2 + x_1 x_2 \\ s.t. \quad & 6x_1 + 2x_2 - 30 = 0 \end{aligned}$$

– Consider now a small change in $\boldsymbol{x} = (x_1, x_2)$:

$$\boldsymbol{x}' = (x_1 + \Delta x_1, x_2 + \Delta x_2)$$

such that the constrained is maintained:

$$g(x_1 + \Delta x_1, x_2 + \Delta x_2) = 0.$$

Then (first-order approximation):

$$\frac{\partial g}{\partial x_1}\Delta x_1 + \frac{\partial g}{\partial x_2}\Delta x_2 = 0$$

This change causes a change in U:

$$\Delta U = \frac{\partial U}{\partial x_1}\Delta x_1 + \frac{\partial U}{\partial x_2}\Delta x_2$$

Combining, we get

$$\begin{aligned}\Delta U &= \Delta x_2 \left(\frac{\partial U}{\partial x_2} - \frac{\partial U}{\partial x_1}\frac{\partial g/\partial x_2}{\partial g/\partial x_1}\right) \\ &\triangleq \Delta x_2 D\end{aligned}$$

Now, if $D > 0$, pick $\Delta x_2 > 0$ or if $D < 0$, pick $\Delta x_2 < 0$.
$\Rightarrow$ at maximum $D = 0$.
Thus,

$$\frac{\partial U/\partial x_1}{\partial U/\partial x_2} = \frac{\partial g/\partial x_1}{\partial g/\partial x_2}$$

or, equivalently,

$$\frac{\partial U/\partial x_1}{\partial g/\partial x_2} = \frac{\partial U/\partial x_1}{\partial g/\partial x_2} = \text{some constant } \lambda$$

– Example:

$$\begin{aligned} \text{maximize} \quad & U(x_1, x_2) \\ s.t. \quad & p_1 x_1 + p_2 x_2 = b \end{aligned}$$

Then, at the maximum

$$\frac{\partial U/\partial x_1}{\partial U/\partial x_2} = \frac{p_1}{p_2}$$

Interpretation: ratio of marginal utilities equals the price ratio at the optimum.

- **The Lagrange Multiplier Method**:

  – Used to solve a problem with equality constraints, e.g.

$$\begin{aligned} \text{maximize} \quad & U(x_1, x_2) \\ s.t. \quad & g(x_1, x_2) = 0 \end{aligned}$$

  – Method: define a new function

$$L = U(x_1, x_2) - \lambda g(x_1, x_2)$$

  and find the unconstrained optimum of this function:

$$\text{Set } \frac{\partial L}{\partial x_1} = 0, \quad \frac{\partial L}{\partial x_2} = 0, \quad \frac{\partial L}{\partial \lambda} = 0$$

  – Why does this work? Set the derivatives to zero as indicated above and see for yourself.

– Example:

$$\begin{array}{rc} \text{maximize} & U(x_1, x_2) = 3x_1 + 4x_2 + x_1x_2 \\ s.t. & 6x_1 + 2x_2 - 30 = 0 \end{array}$$

The Lagrangean function for this problem is:

$$L = 3x_1 + 4x_2 + x_1x_2 - \lambda(6x_1 + 2x_2 - 30)$$

To solve, set each derivative to zero:

$$\frac{\partial L}{\partial x_1} = 0 \quad \Rightarrow \quad 3 + x_2 - 6\lambda = 0$$
$$\frac{\partial L}{\partial x_2} = 0 \quad \Rightarrow \quad 4 + x_1 - 2\lambda = 0$$
$$\frac{\partial L}{\partial \lambda} = 0 \quad \Rightarrow \quad 6x_1 + 2x_2 - 30 = 0$$

Solution (3 equations, 3 unknowns): $x_1 = 1, x_2 = 12$.

Note: it is common to write the optimal solution as $\boldsymbol{x}^* = (x_1^*, x_2^*) = (1, 12)$.

1.3 ECON 101: Introduction

- **What is Economics?**
  Formal study of resource allocation, individual vs. social objectives, mechanisms and institutions, models and policy.

- **Areas of economics**:
  - *Microeconomics*: (detailed picture of individuals)
    * Theory of economics at the level of individual agents, e.g. individual consumers, firms, investors.
    * General theory assumes rational behavior (selfish optimization) and perfect information.
  - *Macroeconomics*: (broad picture)
    * Theory (and practice) of aggregates, e.g., inflation, unemployment, GDP.
  - Other specialized areas:
    * Institutions: central bank, stock exchange.
    * Government policy: taxation, social and infrastructure spending, monetary policy.
    * International economics: trade, protection, tariffs.
    * Development economics: economic paradigms for developing nations.

- **Key elements of microeconomics**:
  - Divisible commodities:
    - Assume a commodity is divisible (real-valued).
    - Let $\boldsymbol{x} = (x_1, \ldots, x_k)$ represent an *allocation* of k commodities:

      $$x_i = \text{ amount of commodity } i$$

      $\boldsymbol{x}$ is called a *commodity bundle*.
  - Analysis of Producers:
    - Cost of production of a commodity.
    - Profit motive and profit maximization.
  - Analysis of Consumers:
    - Preferences amongst commodities.
    - Utility functions and utility maximization.
  - Market economy:
    - Exchange economies and prices.
    - **Def**: **Market**. A collection of consumers and/or producers and a mechanism for trade.
    - **Def**: **Perfect market**. Agents (consumers, producers) are rational, have perfect information and engage in unrestricted trade. Also, no agent is powerful enough to influence prices in isolation.

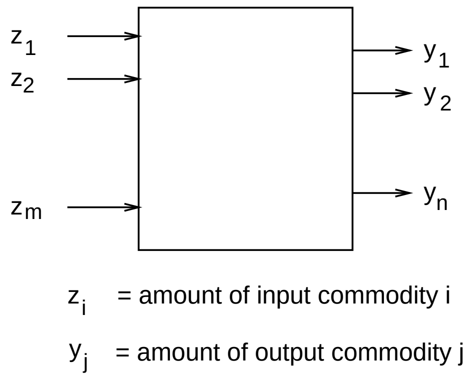
1.4 ECON 101: The Producer

- Think of a producer as a firm manufacturing one or more commodities.

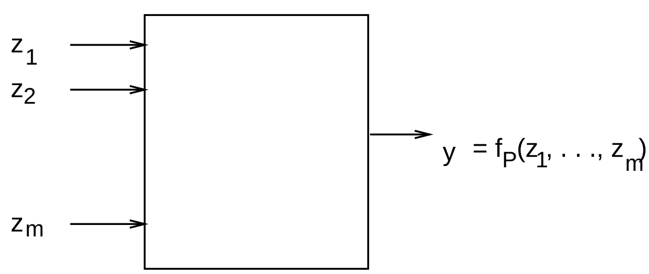
  Goal: a simple mathematical characterization of a producer.

- **Modeling a producer using a** *production function*::

  – The producer is a function from input commodities to output commodities:



  – Typical assumption (for simplicity): only one output



  – Another assumption: analysis is static (for a fixed period of time).

– Example:

$$y = f_p(z_1, z_2) = \gamma\sqrt{z_1}\sqrt{z_2}$$

where

$$\begin{aligned} y &= \text{wheat produced} \\ z_1 &= \text{labor} \\ z_2 &= \text{fertilizer/pesticide} \end{aligned}$$

- **The** ***cost function*** **of a producer**:
  - Suppose a producer uses inputs $\boldsymbol{z} = (z_1, \ldots, z_m)$.
  - If the unit price for commodity z_i is p_i.
    $\Rightarrow$ producer's cost is $p_1 z_1 + \ldots + p_m z_m = \boldsymbol{p} \cdot \boldsymbol{z}$.
  - Suppose the desired level of output is y.
    Then, to minimize costs, the producer solves the following optimization problem:

$$\begin{aligned} \text{minimize} \quad & p_1 z_1 + \ldots + p_m z_m \\ s.t. \quad & f_p(z_1, \ldots, z_m) = y \end{aligned}$$

  - Example: $f_P(z_1, z_2) = \sqrt{z_1}\sqrt{z_2}$, and the desired output level is k units
    $\Rightarrow$ the following problem is solved

$$\begin{aligned} \text{minimize} \quad & p_1 z_1 + p_2 z_2 \\ s.t. \quad & \sqrt{z_1}\sqrt{z_2} = k \end{aligned}$$

    Solve this to get $\boldsymbol{z}^* = (z_1^*, z_2^*) = (k\sqrt{\frac{p_2}{p_1}}, k\sqrt{\frac{p_1}{p_2}})$.

- The *cost function* of the producer is the cost of the optimal solution to the above problem:

$$C(\boldsymbol{p}, y) = \min_{\boldsymbol{z}} \boldsymbol{p} \cdot \boldsymbol{z} \quad s.t. \;\; f_p(\boldsymbol{z}) = y$$

  In the above example,

$$C(p_1, p_2, k) = p_1 k \sqrt{\frac{p_2}{p_1}} + p_2 k \sqrt{\frac{p_1}{p_2}}$$

- Properties of the cost function:
  * Linearity in prices: $C(\alpha \boldsymbol{p}, y) = \alpha C(\boldsymbol{p}, y)$.
  * $C(\boldsymbol{p}, y)$ is non-decreasing in $\boldsymbol{p}$.
  * $C(\boldsymbol{p}, y)$ is concave in $\boldsymbol{p}$.

• **Producers in the presence of consumers**:

- So far, producers have been studied in isolation.
  Now we consider demand and profit maximization.
- The *demand function*:
  * Generally, if a product's price goes up, the demand for that product goes down.
  * Let $D(p)$ = demand seen by a producer at price p.
  * D is called the demand function.
- The *inverse demand function*:
  * Assume D is invertible: $p = D^{-1}(d)$ is called the inverse-demand function, written as $p = P(d)$, for a particular demand value d. It answers the question: what do I set the price to be to create the demand d?
- Maximizing profit:
  * Assume y units are made and all units are sold
    $\Rightarrow$ revenue $R(y) = y \times$ price per unit.
    But, at production level y, the price should be set at

$$P(y) = D^{-1}(y)$$

At this price, revenue is

$$R(y) = yP(y)$$

  * If $C(y)$ is the cost of producing y units (from cost function)

$$\text{profit } \pi(y) = R(y) - C(y) = yP(y) - C(y)$$

- The *basic law of production*:
  * To maximize profit, set derivative of $\pi(y)$ to zero and obtain

$$\frac{dR(y)}{dy} = \frac{dC(y)}{dy}.$$

  * **Basic law of production**: *at maximum profit, marginal revenue equals marginal cost.*
  * Interpretation: if additional revenue obtained from a slight increase in production exceeds cost increase
    $\Rightarrow$ you would produce more.

- **Producers in competition**:
  - Currently, we think of a producer maximizing profit by determining output level y to maximize profit $\pi(y)$.
    $\Rightarrow$ assumes a firm can determine prices via $P(y)$ function
  - In *perfect competition*, a firm cannot determine prices
    $\Rightarrow$ firms are price-takers
    (Any firm that increases prices will not be able to sell anything).
  - If prices are given,

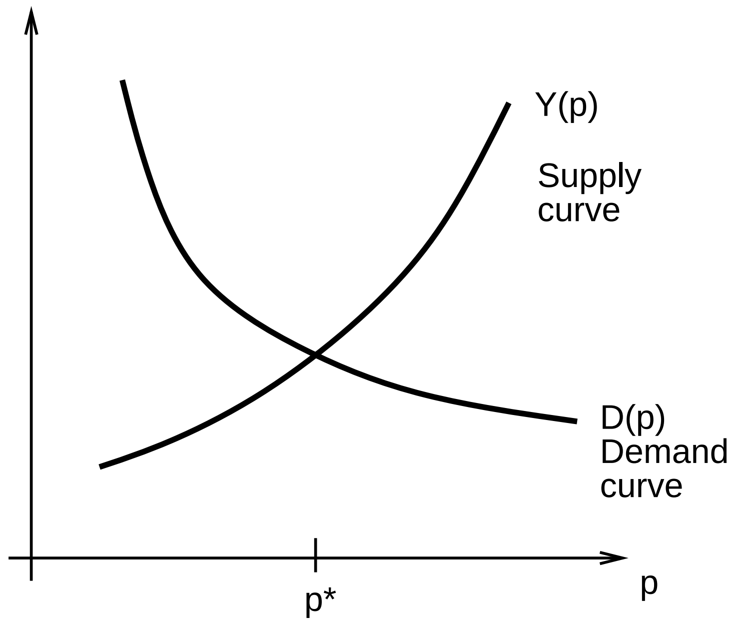
$$\pi(y) = yp - C(y).$$

  - Let $Y(p)$ the output level that maximizes $\pi(y)$ for given price p.
  - $Y(p)$ is called the *supply function*.
  - Usually, in competition, each firm i has a supply function $Y_i(p)$.
    Then, $Y(p) = \sum_i Y_i(p)$ is called the aggregate supply function.

– Generally, $Y(p)$ increases with p.

– Generally, $D(p)$ decreases with p (Demand decreases with increasing price).

– Equilibrium is obtained when $D(p) = Y(p)$
(Supply = demand).



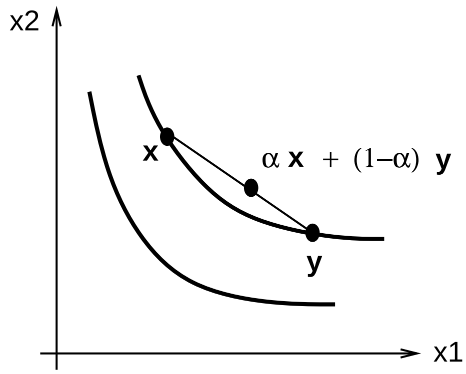
1.5 ECON 101: The Consumer

- **The consumer in isolation: modeling via preferences**
  - How do we model consumer behavior?
    One approach: define a preference relation.
  - Notation: suppose $\boldsymbol{x} = (x_1, \ldots, x_k)$ and $\boldsymbol{y} = (y_1, \ldots, y_k)$ are commodity bundles.
    Example:

(100,20)	$\Rightarrow$	100 units of beer, 20 units of chips
(40,30)	$\Rightarrow$	40 units of beer, 30 units of chips

    Which is a better combination?
  - Each consumer has a preference relation that allows commodity bundles to be compared:
    $\boldsymbol{x} \preceq \boldsymbol{y} \Rightarrow \boldsymbol{y}$ is at least as good as $\boldsymbol{x}$.
  - Define the *indifference* relation $\boldsymbol{x} \approx \boldsymbol{y}$ if both $\boldsymbol{x} \preceq \boldsymbol{y}$ and $\boldsymbol{y} \preceq \boldsymbol{x}$.
    Define $C_{\boldsymbol{x}} \triangleq \{\boldsymbol{y} : \boldsymbol{x} \approx \boldsymbol{y}\}$.
    $C_{\boldsymbol{x}}$ is called an *indifference curve* – the locus of points with equivalent preference.
  - Desirable properties of preference relations:
    - *Continuity*: continuous indifference curves.
    - *Monotonicity*: if $\boldsymbol{x} \geq \boldsymbol{y}$, then $\boldsymbol{x} \succeq \boldsymbol{y}$.
      (More is better).

* *Convexity*: if $\boldsymbol{x} \succeq \boldsymbol{z}$ and $\boldsymbol{y} \succeq \boldsymbol{z}$ then $\alpha \boldsymbol{x} + (1-\alpha)\boldsymbol{y} \succeq \boldsymbol{z}$. $(0 \le \alpha \le 1)$.
  (Indifference curves are convex)

  Interpretation: a balanced allocation is better than an extreme one.

- **The consumer in isolation: modeling via utility functions**
  - Motivation: it's easier to work with functions than relations.
  - **Def**: A utility function is a function U s.t.

$$U(\boldsymbol{x}) \ge U(\boldsymbol{y}) \Leftrightarrow \boldsymbol{x} \succeq \boldsymbol{y}.$$

  - Desirable properties of utility functions:
    * U should be continuous.
    * U should be monotonic:
      $\boldsymbol{x} \ge \boldsymbol{y} \Leftrightarrow U(\boldsymbol{x}) \ge U(\boldsymbol{y})$. (More is better).
    * U should be quasi-concave:
      $U(\boldsymbol{x}) \ge c$ and $U(\boldsymbol{y}) \ge c \Leftrightarrow U(\alpha\boldsymbol{x} + (1-\alpha)\boldsymbol{y}) \ge c$.
  - **Fact**: if the preference relation is continuous and monotonic, then a continuous utility function exists.
  - **Fact**: U quasi-concave $\Leftrightarrow$ convex indifference curves.

- **The consumer and prices**:
  - Suppose commodity i has unit price p_i.

- Assume consumer has budget b.
- The consumer's behavior is modeled as:

$$\begin{array}{rl} \text{maximize} & U(\boldsymbol{x}) \\ s.t. & \sum_i p_i x_i \leq b. \end{array}$$

- Example:

$$\begin{aligned} U(x_1, x_2) &= 8x_1 + 4x_2 + 2x_1x_2 \\ p_1 &= 5 \\ p_2 &= 10 \\ b &= 100 \end{aligned}$$

  Hence the consumer solves

$$\begin{array}{rl} \text{maximize} & 8x_1 + 4x_2 + 2x_1x_2 \\ s.t. & 5x_1 + 10x_2 = 100 \end{array}$$

  Solution turns out to be: $x_1^* = 13.25, x_2^* = 4.375$.

- **The consumer's demand function**:
  - Consider a consumer with utility function U and budget b.
  - We assume the consumer solves

$$\begin{array}{rl} \text{maximize} & U(\boldsymbol{x}) \\ s.t. & \sum_i p_i x_i \leq b \end{array}$$

    The solution depends on both $\boldsymbol{p}$ and b.
  - Let $X_U(\boldsymbol{p}, b)$ be the solution you get for prices $\boldsymbol{p}$ and budget b.
  - $X_U(\boldsymbol{p}, b)$ is called the *demand function*.
  - Generally, $X_U(\boldsymbol{p}, b)$ decreases with increasing $\boldsymbol{p}$.

1.6 ECON 101: An Exchange Economy

- **What is an exchange economy?**
  - Consider a market with many consumers and NO producers.
  - Assume k commodities.
  - Consumer i has initial endowment $\boldsymbol{x}^i = (x_1^i, \ldots, x_k^i)$, i.e., x_l^i amount of commodity l.
  - What kind of economic activity is possible without producers?
    $\Rightarrow$ consumers can trade
    $\Rightarrow$ called an *exchange economy*

- **Example**: consider two consumers (A and B) and two commodities
  - Suppose consumer A has utility function $U_A(x_1^A, x_2^A) = 4x_1^A + 2x_2^A$ and initial endowment (4,12).
    $\Rightarrow$ initial utility of A $= U_A(4, 12) = 40$.
  - Suppose consumer B has utility function $U_B(x_1^B, x_2^B) = x_1^B + 2x_2^B$ and initial endowment (16,8).
    $\Rightarrow$ initial utility of B $= U_B(16, 8) = 32$.
  - Note: $x_1^A + x_1^B = 20$ and $x_2^A + x_2^B = 20$.
  - Suppose A and B consider trading so that
    - A gets (12,4) and
    - B gets (8,16).

    What are the new utilities? Should they trade?
  - Suppose A and B consider trading so that
    - A gets (6,8) and
    - B gets (14,12).

What are the new utilities? Should they trade?

- Assumption: a trade will occur if at least one improves without hurting the other.
- What about when
  * A gets (8,16) and
  * B gets (12,4)?
- Suppose the initial endowments were such that
  * A has (20,20) and
  * B has (0,0).

  There is *no* trade that A could possibly agree to.

• **Def**: An allocation is *Pareto-optimal* if there is no trade that improves the utility of at least one guy while not hurting the others.

• NOTE:

- Pareto-optimality is a way of evaluating the value of an allocation.
- An alternative way is to define a society-wide utility function in terms of individual utility functions.
  e.g. $U_S(\boldsymbol{x}^A, \boldsymbol{x}^B) = U_A(\boldsymbol{x}^A) + U_B(\boldsymbol{x}^B)$.
- In this case, the "best" allocation is the one that minimizes U_S.
- However, the sum-of-utilities is somewhat artificial
  ⇒ product-of-utilities will give a different result.
- Pareto-optimality instead uses only individual utilities and considers allocations from a selfish-individual perspective.

- **2 consumers and 2 commodities: some analysis**
  - Suppose
    * Consumer A has initial endowment $(\bar{x}_1^A, \bar{x}_2^A)$.
    * Consumer B has initial endowment $(\bar{x}_1^B, \bar{x}_2^B)$.

    with totals $S_1 = \bar{x}_1^A + \bar{x}_1^B$ and $S_2 = \bar{x}_2^A + \bar{x}_2^B$.
  - To achieve Pareto-optimality, consumer A solves

$$\begin{aligned} \text{maximize} \quad & U_A(x_1^A, x_2^A) \\ s.t. \quad & x_1^A + x_1^B = S_1 \\ & x_2^A + x_2^B = S_2 \\ & U_B(x_1^B, x_2^B) \geq U_B(\bar{x}_1^B, \bar{x}_2^B) \end{aligned}$$

    (Consumer B solves a similar problem).

    The Lagrange Multiplier Method is used to obtain:

$$\frac{\partial U_A/\partial x_1^A}{\partial U_A/\partial x_2^A} = \frac{\partial U_B/\partial x_1^B}{\partial U_B/\partial x_2^B}$$

  - Interpretation: at the optimum, marginal rates of substitution are the same across all individuals.

    If they are not, a small trade would improve both individuals' utilities.

1.7 ECON 101: Competitive Equilibrium and the Fundamental Theorems of Economics

- Consider an exchange economy with 2 commodities and 2 consumers:
  - Suppose the individuals cannot affect prices in isolation (say, an external referee selects prices).
  - Let the prices be $\boldsymbol{p} = (p_1, p_2)$ for the two goods.
  - Suppose
    * Consumer A has initial endowment $(\bar{x}_1^A, \bar{x}_2^A)$.
    * Consumer B has initial endowment $(\bar{x}_1^B, \bar{x}_2^B)$.

    Then,
    * A's initial budget (worth) is $b_A = p_1\bar{x}_1^A + p_2\bar{x}_2^A$.
    * B's initial budget (worth) is $b_B = p_1\bar{x}_1^B + p_2\bar{x}_2^B$.
  - **Def**: The price-allocation combination of

$$\begin{aligned} \boldsymbol{x}^A &= (x_1^A, x_2^A) \\ \boldsymbol{x}^B &= (x_1^B, x_2^B) \\ \boldsymbol{p} &= (p_1, p_2) \end{aligned}$$

    is called a *competitive equilibrium* if

    1. the allocations are feasible, i.e.,

$$\begin{aligned} x_1^A + x_1^B &= \bar{x}_1^A + \bar{x}_1^B \\ x_2^A + x_2^B &= \bar{x}_2^A + \bar{x}_2^B \end{aligned}$$

2. the allocations are budget-feasible, i.e.,

$$\begin{aligned} p_1 x_1^A + p_2 x_2^A &= b_A \\ p_1 x_1^B + p_2 x_2^B &= b_B \end{aligned}$$

3. for every other budget-feasible allocation $\boldsymbol{y}^A$, $\boldsymbol{y}^B$, the following is true:

$$\begin{aligned} U_A(\boldsymbol{y}^A) &\leq U_A(\boldsymbol{x}^A) \\ U_B(\boldsymbol{y}^B) &\leq U_B(\boldsymbol{x}^B) \end{aligned}$$

- **First Fundamental Theorem of (Welfare) Economics**:
  A price-allocation combination $(\boldsymbol{x}^A, \boldsymbol{x}^B, \boldsymbol{p})$ that satisfies competitive equilibrium is Pareto-optimal.
  (Technical assumptions need to be made, e.g., quasi-concave utility functions).

- **Second Fundamental Theorem of (Welfare) Economics**:
  If the allocation $\boldsymbol{x}^A$, $\boldsymbol{x}^B$ is Pareto-optimal, then there is a price vector $\boldsymbol{p}$ such that $(\boldsymbol{x}^A, \boldsymbol{x}^B, \boldsymbol{p})$ is a competitive equilibrium.
  (Similar technical assumptions).

- NOTE:
  - Pareto-optimality is not defined in terms of prices.
    The theorems ensure that linear (per-unit) pricing allows one to achieve Pareto-optimality through price-constrained selfish optimization.
  - The result holds for multiple commodities and consumers.
  - A more general result includes the presence of producers.

1.8 Tatonnement

- So far, we only only discussed the *existence* of equilibrium prices.

  Key question: how to implement a mechanism to find the equilibrium prices (and hence, a Pareto-optimal allocation)?

- **Omniscient dictatorship approach**:
  - A dictator uses the necessary conditions for Pareto-optimality and *computes* the equilibrium prices and allocations.
  - Consumers are informed of their optimal allocations.
  - Consumers then make exchanges to achieve the optimal allocation.

  Drawback of this method: presumes the existence of an omniscient dictatorship.

- **Tatonnement**: a decentralized implementation of price-determination.

- **Key ideas in Tatonnement**:
  - The referee selects arbitrary an initial price for each commodity.
  - Each consumer maximizes his/her utility within budget constraints to obtain his/her desired allocation.
  - Each consumer reports his/her desired allocation to a referee.
  - The referee looks at the total amount requested for each commodity:
    * If the total amount requested is *more* than the total available, the referee *increases the price* for that commodity.
    * If the total amount requested is *less* than the total available, the referee *decreases the price* for that commodity.
    * Otherwise, price is unchanged.
  - If any one price changed, the referee reports the new prices to the consumers.

- The process is repeated until prices have converged.
- The limiting prices are taken as the equilibrium prices.
- Finally, the allocations are determined based on these equilibrium prices.

NOTE:

- Utility computations are decentralized (each consumer computes his/her own utility maximization).
- High demand for a commodity increases its price.
- Low demand decreases the price.

1.9 Tatonnement as an Algorithm

- **Observation**: Tatonnement can be used as a distributed algorithm for resource allocation in a distributed system.
- **Example**:
  - Suppose 2 branches of a bank wish to share 2 database files.
    $\Rightarrow$ 2 commodities (the 2 files) and 2 consumers (the 2 branches).
  - Suppose the files are accessed by queries generated locally.
    * Some queries generated at A only access File 1, others access only File 2.
    * Some queries generated at B access both files.
    * The same holds for queries generated at B.
  - Suppose the following probabilities are known (via estimation, say)

$$\begin{aligned} \alpha_1^A &= P\,[\text{An access at A is only for File 1}] \\ \alpha_2^A &= P\,[\text{An access at A is only for File 2}] \\ \alpha_{12}^A &= P\,[\text{An access at A is for both files}] \end{aligned}$$

    Here, $\alpha_1^A + \alpha_2^A + \alpha_{12}^A = 1$.
  - Similarly, define

$$\begin{aligned} \alpha_1^B &= P\,[\text{An access at B is only for File 1}] \\ \alpha_2^B &= P\,[\text{An access at B is only for File 2}] \\ \alpha_{12}^B &= P\,[\text{An access at B is for both files}] \end{aligned}$$

  - Next, assume the files are divisible:

* x_1^A = fraction of File 1 stored at A.
* x_2^A = fraction of File 2 stored at A.

– Each consumer would like to maximize the probability that an access is locally satisfied.

$$P\,[\text{An access at A is locally satisfied}] = \alpha_1^A x_1^A + \alpha_2^A x_2^A + \alpha_{12}^A x_1^A x_2^A$$

(Similar expression for B).

– Thus, the utility functions of the consumers are:

$$\begin{aligned} U_A(x_1^A, x_2^A) &= \alpha_1^A x_1^A + \alpha_2^A x_2^A + \alpha_{12}^A x_1^A x_2^A \\ U_B(x_1^B, x_2^B) &= \alpha_1^B x_1^B + \alpha_2^B x_2^B + \alpha_{12}^B x_1^B x_2^B \end{aligned}$$

– Thus, given prices (p_1, p_2), consumer A solves

$$\begin{aligned} \text{maximize} \quad & \alpha_1^A x_1^A + \alpha_2^A x_2^A + \alpha_{12}^A x_1^A x_2^A \\ s.t. \quad & p_1 x_1^A + p_2 x_2^A \le b_A \\ & 0 \le x_1^A \le 1, \quad 0 \le x_2^A \le 1. \end{aligned}$$

Using the Lagrange Multiplier Method, consumer A obtains

$$\begin{aligned} x_1^A(p_1, p_2) &= \frac{\alpha_{12}^A b_A + \alpha_1^A p_2 - \alpha_2^A p_1}{2\alpha_{12}^A p_1} \\ x_2^A(p_1, p_2) &= \frac{\alpha_{12}^A b_A + \alpha_2^A p_1 - \alpha_1^A p_2}{2\alpha_{12}^A p_2} \end{aligned}$$

(Consumer B solves a similar problem)

- The basic algorithm:

Algorithm: TATONNEMENT()

```
for i := 1 to num_commodities
    p[i] := initial price of i-th commodity;
    total_amount[i] := total amount of i-th commodity;
endfor
for n := 1 to num_iterations
    // Use current prices p to maximize utility
    Compute amounts x^A[i] that maximize A's utility;
    Compute amounts x^B[i] that maximize B's utility;
    for i := 1 to num_commodities
        total_demand[i] := x^A[i] + x^B[i];
        p[i] := p[i] + η (total_amount[i] - total_demand[i]);
    endfor
endfor
```

- Another example: how to finesse the handling of indivisible commodities
  - Suppose now that each of A and B above also want to share a printer.
  - How is a printer to be divided?
  - One way of "sharing" a printer:
    - * Whenever A sends a print job to the queue, a coin with $P[\text{heads}] = y^A$ is flipped.
      If *heads* is obtained, all of A's jobs are moved to the head of the queue.
      If *tails* is obtained, A's new job joins the end of the print queue.
    - * A similar y^B-biased coin is associated with B.
    - * We will enforce $y^A + y^B = 1$
      (even though it's enough to ensure $y^A + y^B = \text{constant}$).
  - Then, along with the two files, A's utility function is

$$U_A(x_1^A, x_2^A) = \alpha_1^A x_1^A + \alpha_2^A x_2^A + \alpha_{12}^A x_1^A x_2^A + \beta^A y^A$$

    where β^A is a constant.
  - Similarly, B's utility function is

$$U_B(x_1^B, x_2^B) = \alpha_1^B x_1^B + \alpha_2^B x_2^B + \alpha_{12}^B x_1^B x_2^B + \beta^B y^B$$