Conclusions

Newton-Raphson Procedure

- Fails to converge for about $\frac{1}{5}$ of cases.
- Each iteration requires more computation than SIP.
- For cases in which it converges, Newton-Raphson normally requires more iterations.

Simple Iterative Procedure

- Always converges to a result for $c^*$.
- Almost always requires less iterations.
- Always requires less computation.
A Comparison

Table 1 contains a comparison of the two procedures.
An Illustration

The figure below illustrates the behavior of the SIP.
The Simple Iterative Procedure

- Finding $c^*$ is now equivalent to finding a fixed point of $q(c)$.
- (McCool) $q(c)$ has only one fixed point, the unique solution $c^*$.

The Simple Iterative Procedure begins with

$$c_{k+1} = \frac{c_k + q(c_k)}{2} \quad (12)$$

starting at an initial point $c_0$ for $k = 0, 1, 2, \ldots$.

From Theorem 2 in the paper, we have the following:

- The SIP always converges for any choice $c_0 > 0$.
- The SIP converges with geometric rate $\frac{1}{2}$.
Creating the SIP

By using the new notation introduced on the previous slide, we can rewrite equation (2) as

\[
\frac{s_3}{s_2} - \frac{1}{c} = \frac{s_1}{n}.
\]  

(9)

Solving the above equation for \(c\), we obtain

\[
\frac{1}{c} = \frac{s_3}{s_2} - \frac{s_1}{n}
\]

\[
c = \frac{ns_2}{ns_3 - s_1s_2}.
\]  

(10)

If we let

\[
q(c) = \frac{ns_2}{ns_3 - s_1s_2}
\]  

(11)

then we have reduced equation (2) to \(q(c) = c\).
A New Look

To simplify our notations, let

\[ s_1(c) = \sum_{i=1}^{n} \log x_i, \quad s_2(c) = \sum_{i=1}^{n} x_i^c, \]  \tag{6}

\[ s_3(c) = \sum_{i=1}^{n} x_i^c \log x_i, \quad \text{and} \quad s_4(c) = \sum_{i=1}^{n} x_i^c \log^2 x_i. \]  \tag{7}

So, we can write the Newton-Raphson procedure as

\[ c_{k+1} = c_k - \frac{c}{n} \frac{s_1 s_2 - cs_3 + s_2}{\frac{1}{n}s_1(s_2 + cs_3) - cs_4}. \]  \tag{8}
Newton-Raphson Method used for Parameter Estimation

\[ g(c) = \frac{c}{n} \sum_{i=1}^{n} \log x_i \sum_{i=1}^{n} x_i^c - c \sum_{i=1}^{n} x_i^c \log x_i + \sum_{i=1}^{n} x_i^c \quad (4) \]

\[ g'(c) = \frac{1}{n} \sum_{i=1}^{n} \log x_i \left( \sum_{i=1}^{n} x_i^c + c \sum_{i=1}^{n} x_i^c \log x_i \right) - c \sum_{i=1}^{n} x_i^c \log^2 x_i \quad (5) \]

- (4) is arrived at by setting equation (2) equal to 0.
- (5) is simply the first derivative of (4).
The Newton-Raphson Method

\[ c_{k+1} = c_k - \frac{g(c_k)}{g'(c_k)}, \quad \text{for } k = 0, 1, 2, \cdots \]  

- The value \( |c_{k+1} - c_k| \) does not always converge to 0.
- Convergence depends upon the choice of the initial value, \( c_0 \).
Maximum Likelihood Estimators

\[ b = \left[ \frac{\sum_{i=1}^{n} x_i^c}{n} \right]^\frac{1}{c} \]  

(1)

\[ \frac{\sum_{i=1}^{n} x_i^c \log x_i}{n} - \frac{1}{c} = \frac{1}{n} \sum_{i=1}^{n} \log x_i \]  

(2)

- (McCool) A unique, positive solution \(c^*\) exists for (2).
- There is no known analytical method for solving (2).
Weibull Distribution

\[ f(x; a, b, c) = \frac{c(x - a)^{c-1}}{bc} \exp \left\{ - \left(\frac{x - a}{b}\right)^c \right\} \]

where \( x \geq a, \ b > 0, \ c > 0 \).

The three parameters are

- \( a \) (location - set to 0 for this paper)
- \( b \) (scale)
- \( c \) (shape)

The Weibull pdf is useful as a failure model in analyzing the reliability of different types of systems.
Parameter estimation of the Weibull probability distribution

- Weibull Distribution
- Newton-Raphson Method
- Application to Weibull Parameter Estimation Problem
- Simple Iterative Procedure (SIP)
- Comparision of Both Procedures