Problem 1. Prove by induction:

$$1 + 4 + 4^2 + ... + 4^n = (4^{n+1} - 1)/3$$

for all $n \in \boldsymbol{Z}$, $n \geq 0$.

Comments on the induction method. Let $P(n)$ be a *formula* with one integer variable n. This means that $P(n)$ can be either *true* or *false* when n takes a value. For example

$$P(n) : 1 + 4 + 4^2 + ... + 4^n = (4^{n+1} - 1)/3.$$

We want to prove the *proposition* $\forall n P(n)$.

1. *Base case.* We prove the *proposition* $P(n_0)$ where n_0 is the smallest value that n can receive. In our example $n_0 = 0$, so we prove the proposition:

$$P(0) : 1 = (4^{0+1} - 1)/3$$

2. *Induction step.* We prove the *proposition* $\forall n(P(n) \Rightarrow P(n+1))$. In our example we have

$$\begin{array}{rcl} P(n) & : & 1 + 4 + 4^2 + ... + 4^n = (4^{n+1} - 1)/3 \\ P(n+1) & : & 1 + 4 + 4^2 + ... + 4^{n+1} = (4^{n+2} - 1)/3 \end{array}$$

so we have to prove that $\forall n \geq 0$

$$\begin{array}{rcl} 1 + 4 + 4^2 + ... + 4^n & = & (4^{n+1} - 1)/3 \Rightarrow \\ 1 + 4 + 4^2 + ... + 4^{n+1} & = & (4^{n+2} - 1)/3. \end{array}$$

Solution.

1. *Base case.* For $n = 0$ we have

$$1 = (4^{0+1} - 1)/3 = 3/3$$

which is true.

2. *Induction step.*

1st method, direct: $P(n) \Rightarrow P(n+1)$.

We have

$$\begin{array}{rcl} \sum_{i=0}^{n} 4^i & = & (4^{n+1} - 1)/3 \Rightarrow \\ \sum_{i=0}^{n} 4^i + 4^{n+1} & = & (4^{n+1} - 1)/3 + 4^{n+1} \Rightarrow \\ \sum_{i=0}^{n+1} 4^i & = & (4^{n+1} - 1)/3 + \frac{3 \cdot 4^{n+1}}{3} \\ & = & \frac{(3+1)4^{n+1} - 1}{3} \\ & = & \frac{4^{n+2} - 1}{3}. \end{array}$$

2nd method We prove $P(n) \Rightarrow P(n+1)$ by proving $P(n+1) \Leftrightarrow P(n)$. We have

$$\begin{aligned}
\sum_{i=0}^{n+1} 4^i &= (4^{n+2}-1)/3 \Leftrightarrow \\
\sum_{i=0}^{n} 4^i + 4^{n+1} &= (4^{n+2}-1)/3 \Leftrightarrow \\
\sum_{i=0}^{n} 4^i &= (4^{n+2}-1)/3 - 4^{n+1} \\
&= (4 \cdot 4^{n+1}-1)/3 - \frac{3 \cdot 4^{n+1}}{3} \\
&= \frac{(4-3)4^{n+1}-1}{3} \\
&= \frac{4^{n+1}-1}{3}.
\end{aligned}$$

3d method, indirect We prove $P(n) \Rightarrow P(n+1)$ by substitution. Assume that

$$\sum_{i=0}^{n} 4^i = (4^{n+1}-1)/3$$

is true. Then

$$\begin{aligned}
\sum_{i=0}^{n+1} 4^i &= (4^{n+2}-1)/3 \Leftrightarrow \\
\sum_{i=0}^{n} 4^i + 4^{n+1} &= (4^{n+2}-1)/3 \Leftrightarrow \\
(4^{n+1}-1)/3 + 4^{n+1} &= (4^{n+2}-1)/3 \Leftrightarrow \\
(4^{n+1}-1)/3 + \frac{3 \cdot 4^{n+1}}{3} &= (4^{n+2}-1)/3 \Leftrightarrow \\
(4 \cdot 4^{n+1}-1)/3 &= (4^{n+2}-1)/3.
\end{aligned}$$

The last equality is true, so the first equality is also true and we have proved that $P(n) \Rightarrow P(n+1)$.