Section 2.3: Monte Carlo Simulation
With *Empirical Probability*, we perform an experiment many times $n$ and count the number of occurrences $n_a$ of an event $\mathcal{A}$.

- The *relative frequency* of occurrence of event $\mathcal{A}$ is $n_a/n$.
- The *frequency theory of probability* asserts that the relative frequency converges as $n \to \infty$.

\[
\Pr(\mathcal{A}) = \lim_{n \to \infty} \frac{n_a}{n}
\]

*Axiomatic Probability* is a formal, set-theoretic approach.

- Mathematically construct the sample space and calculate the number of events $\mathcal{A}$.
- The two are complementary!
Example 2.3.1

- Roll two dice and observe the up faces
  
  \[(1, 1) \ (1, 2) \ (1, 3) \ (1, 4) \ (1, 5) \ (1, 6)\]
  \[(2, 1) \ (2, 2) \ (2, 3) \ (2, 4) \ (2, 5) \ (2, 6)\]
  \[(3, 1) \ (3, 2) \ (3, 3) \ (3, 4) \ (3, 5) \ (3, 6)\]
  \[(4, 1) \ (4, 2) \ (4, 3) \ (4, 4) \ (4, 5) \ (4, 6)\]
  \[(5, 1) \ (5, 2) \ (5, 3) \ (5, 4) \ (5, 5) \ (5, 6)\]
  \[(6, 1) \ (6, 2) \ (6, 3) \ (6, 4) \ (6, 5) \ (6, 6)\]

- If the two up faces are summed, an integer-valued random variable, say \(X\), is defined with possible values 2 through 12 inclusive

  \[
  \begin{array}{c}
  \text{sum, } x : & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 & 12 \\
  \Pr(X = x) : & \frac{1}{36} & \frac{2}{36} & \frac{3}{36} & \frac{4}{36} & \frac{5}{36} & \frac{6}{36} & \frac{5}{36} & \frac{4}{36} & \frac{3}{36} & \frac{2}{36} & \frac{1}{36}
  \end{array}
  \]

- \(\Pr(X = 7)\) could be estimated by replicating the experiment many times and calculating the relative frequency of occurrence of 7’s
A Random Variate is an algorithmically generated realization of a random variable. $u = \text{Random}()$ generates a Uniform(0, 1) random variate. How can we generate a Uniform($a$, $b$) variate?

$$x = a + (b - a)u$$

Generating a Uniform Random Variate

```cpp
double Uniform(double a, double b) /* use $a < b$ */ {
    return (a + (b - a) * Random());
}
```
Equilikely Random Variates

- *Uniform*(0, 1) random variates can also be used to generate an *Equilikely*(a, b) random variate

\[
0 < u < 1 \iff 0 < (b - a + 1)u < b - a + 1
\]
\[
\iff 0 \leq \lfloor (b - a + 1)u \rfloor \leq b - a
\]
\[
\iff a \leq a + \lfloor (b - a + 1)u \rfloor \leq b
\]
\[
\iff a \leq x \leq b
\]

Specifically, \( x = a + \lfloor (b - a + 1)u \rfloor \)

Generating an Equilikely Random Variate

```c
long Equilike\(\text{ly}(long\ a,\ long\ b)\) /* use a < b */ {
    return (a + (long) ((b - a + 1) * Random()));
}
```
Example 2.3.3 To generate a random variate $x$ that simulates rolling two fair dice and summing the resulting up faces, use $x = \text{Equil Likely}(1, 6) + \text{Equil Likely}(1, 6)$; Note that this is not equivalent to $x = \text{Equil Likely}(2, 12)$;

Example 2.3.4 To select an element $x$ at random from the array $a[0], a[1], \ldots, a[n-1]$ use $i = \text{Equil Likely}(0, n - 1); x = a[i]$;
If three fair dice are rolled, which sum is more likely, a 9 or a 10?

There are $6^3 = 216$ possible outcomes.

$$\Pr(X = 9) = \frac{25}{216} \approx 0.116 \quad \text{and} \quad \Pr(X = 10) = \frac{27}{216} = 0.125$$

Program galileo calculates the probability of each possible sum between 3 and 18.

The drawback of Monte Carlo simulation is that it only produces an estimate.

Larger $n$ does not guarantee a more accurate estimate.
Example 2.3.6

- Frequency probability estimates converge slowly and somewhat erratically

You should *always* run a Monte Carlo simulation with multiple initial seeds
Generate a point at random inside a rectangle with opposite corners at \((\alpha_1, \beta_1)\) and \((\alpha_2, \beta_2)\)

\[x = \text{Uniform}(\alpha_1, \alpha_2); \quad y = \text{Uniform}(\beta_1, \beta_2);\]
• Generate a point \((x, y)\) at random on the circumference of a circle with radius \(\rho\) and center \((\alpha, \beta)\)

\[
\begin{align*}
\theta &= \text{Uniform}(-\pi, \pi) ; \\
x &= \alpha + \rho \times \cos(\theta) ; \\
y &= \beta + \rho \times \sin(\theta) ;
\end{align*}
\]
Example 2.3.8

- Generate a point \((x, y)\) at random \textit{interior} to the circle of radius \(\rho\) centered at \((\alpha, \beta)\)

\[\begin{align*}
\theta &= \text{Uniform}(\!-\pi, \pi)\;;
\quad r = \text{Uniform}(0, \rho)\;;
\quad \text{INCORRECT!} \quad x &= \alpha + r \times \cos(\theta)\;;
\quad y = \beta + r \times \sin(\theta)\;;
\end{align*}\]
Acceptance/Rejection

- Generate a point at random within a circumscribed square and then either accept or reject the point

**Generating a Random Point**

```plaintext
do {
    x = Uniform(-\rho, \rho);
    y = Uniform(-\rho, \rho);
} while (x * x + y * y >= \rho * \rho);

x = \alpha + x;
y = \beta + y;
return (x, y);
```
Suppose that an infinite family of infinitely long vertical lines are spaced one unit apart in the \((x, y)\) plane. If a needle of length \(r > 0\) is dropped at random onto the plane, what is the probability that it will land crossing at least one line?

- \(u\) is the \(x\)-coordinate of the left-hand endpoint
- \(v\) is the \(x\)-coordinate of the right-hand endpoint, \(v = u + r \cos \theta\)
- The needle crosses at least one line if and only if \(v > 1\)
Program buffon

- Program buffon is a Monte Carlo simulation
  - The random number library can be used to automatically generate an initial seed

#### Random Seeding

```c
PutSeed(-1); /* any negative integer will do */
GetSeed(&seed); /* trap the value of the initial seed */

printf("with an initial seed of %ld", seed);
```

- Inspection of the program buffon illustrates how to solve the problem axiomatically
“Dropped at random” is interpreted (modeled) to mean that $u$ and $\theta$ are independent $Uniform(0, 1)$ and $Uniform(-\pi/2, \pi/2)$ r.v.s.
The shaded region has a curved boundary defined by the equation $u = 1 - r \cos(\theta)$.

If $0 < r \leq 1$, the area of the shaded region is

$$\pi - \int_{-\pi/2}^{\pi/2} (1 - r \cos \theta) \, d\theta = r \int_{-\pi/2}^{\pi/2} \cos \theta \, d\theta = \cdots = 2r$$

Therefore, because the area of the rectangle is $\pi$ the probability that the needle will cross at least one line is $2r/\pi$