Discrete-Event Simulation:

A First Course

Section 4.2: Discrete-Data Histograms

Section 4.2: Discrete-Data Histograms

- Given a discrete-data sample multiset $\mathcal{S} = \{x_1, x_2, \ldots, x_n\}$ with possible values $\mathcal{X}$, the *relative frequency* is

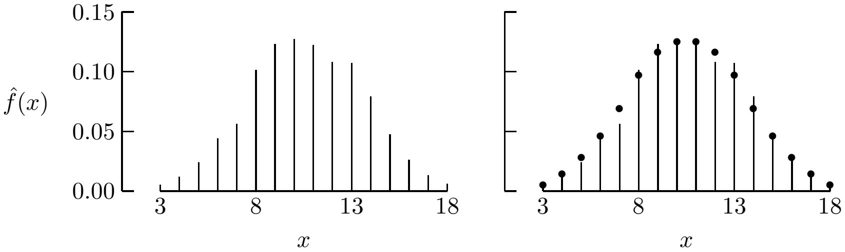
$$\hat{f}(x) = \frac{\text{the number of } x_i \in \mathcal{S} \text{ with } x_i = x}{n}$$

- A *discrete-data histogram* is a graphical display of $\hat{f}(x)$ versus x
- If $n = |\mathcal{S}|$ is large relative to $|\mathcal{X}|$ then values will appear multiple times

Example 4.2.1

Program `galileo` was used to replicate $n = 1000$ rolls of three dice

- Discrete-data sample is $\mathcal{S} = \{x_1, x_2, \ldots, x_{1000}\}$
- Each x_i is an integer between 3 and 18, so $\mathcal{X} = \{3, 4, \ldots, 18\}$
- Theoretical probabilities: $\bullet$'s (limit as $n \to \infty$)



- Since $\mathcal{X}$ is known *a priori*, use an array

Example 4.2.2

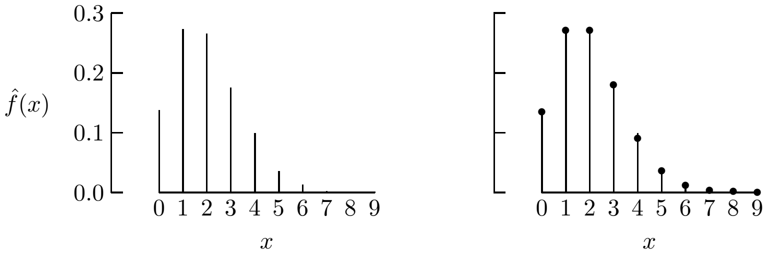
Suppose $2n = 2000$ balls are placed *at random* into $n = 1000$ boxes:

Example 4.2.2

```
n = 1000;
for (i = 1; i<= n; i++)      /* i counts boxes */
    x_i = 0;
for (j = 1; i<= 2*n; i++) {      /* j counts balls */
    i = Equilikely(1, n);      /* pick a box at random */
    x_i++;          /* then put a ball in it */
}
return x_1, x_2, ..., x_n;
```

Example 4.2.2

- $\mathcal{S} = \{x_1, x_2, \ldots, x_n\}$, x_i is the number of balls placed in box i
- $\bar{x} = 2.0$
- Some boxes will be empty, some will have one ball, some will have two balls, etc.
- For seed 12345:



Histogram Mean and Standard Deviation

- The *discrete-data histogram mean* is

$$\bar{x} = \sum_x x\hat{f}(x)$$

- The *discrete-data histogram standard deviation* is

$$s = \sqrt{\sum_x (x - \bar{x})^2 \hat{f}(x)} \quad \text{or} \quad s = \sqrt{\left(\sum_x x^2 \hat{f}(x)\right) - \bar{x}^2}$$

- The *discrete-data histogram variance* is s^2

Histogram Mean versus Sample Mean

- By definition, $\hat{f}(x) \geq 0$ for all $x \in \mathcal{X}$ and

$$\sum_{x} \hat{f}(x) = 1$$

- From the definition of $\mathcal{S}$ and $\mathcal{X}$,

$$\sum_{i=1}^{n} x_i = \sum_{x} x n \hat{f}(x) \qquad \text{and} \qquad \sum_{i=1}^{n} (x_i - \bar{x})^2 = \sum_{x} (x - \bar{x})^2 n \hat{f}(x)$$

- The *sample* mean/standard deviation is mathematically equivalent to the *discrete-data histogram* mean/standard deviation
- If frequencies $\hat{f}(\cdot)$ have already been computed, $\bar{x}$ and s should be computed using discrete-data histogram equations

Example 4.2.3

- For the data in Example 4.2.1 (three dice):

$$\bar{x} = \sum_{x=3}^{18} x\hat{f}(x) \cong 10.609 \qquad \text{and} \qquad s = \sqrt{\sum_{x=3}^{18}(x-\bar{x})^2\hat{f}(x)} \cong 2.925$$

- For the data in Example 4.2.2 (balls placed in boxes)

$$\bar{x} = \sum_{x=0}^{9} x\hat{f}(x) = 2.0 \qquad \text{and} \qquad s = \sqrt{\sum_{x=0}^{9}(x-\bar{x})^2\hat{f}(x)} \cong 1.419$$

Algorithm 4.2.1

Given integers a, b and integer-valued data $x_1, x_2, \ldots$ the following computes a discrete-data histogram:

Algorithm 4.2.1

```
long count[b-a+1];
n = 0;
for (x = a; x <= b; x++)
    count[x-a] = 0;
outliers.lo = 0;
outliers.hi = 0;
while ( more data ) {
    x = GetData();
    n++;
    if ((a <= x) and (x <= b))
        count[x-a]++;
    else if (a > x)
        outliers.lo++;
    else
        outliers.hi++;
}
return n, count[], outliers /* f̂(x) is (count[x-a] / n) */
```

Outliers

- Algorithm 4.2.1 allows for *outliers*
    - Occasional x_i outside the range $a \leq x_i \leq b$
    - Necessary due to array structure
- Outliers are common with some experimentally measured data
- Generally, *valid* discrete-event simulations should not produce *any* outlier data

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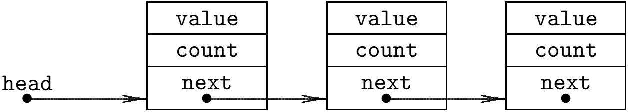
General-Purpose Discrete-Data Histogram Algorithm

Algorithm 4.2.1 is not a good *general purpose* algorithm:

- For integer-valued data, a and b must be chosen properly, or else
  - Outliers may be produced without justification
  - The `count` array may needlessly require excessive memory
- For data that is not integer-valued, algorithm 4.2.1 is not applicable

We will use a linked-list histogram

Linked-List Discrete-Data Histograms

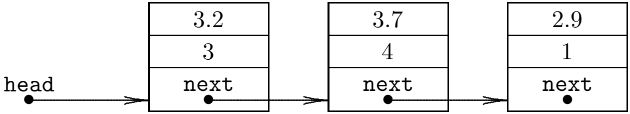


Algorithm 4.2.2

- Initialize the first list node, where `value` $= x_1$ and `count` $= 1$
- For all sample data x_i, $i = 2, 3, \ldots, n$
  - Traverse the list to find a node with `value` $== x_i$
  - If found, increase corresponding `count` by one
  - Else add a new node, with `value` $= x_i$ and `count` $= 1$

Example 4.2.4

- Discrete data sample $\mathcal{S} = \{3.2, 3.7, 3.7, 2.9, 3.7, 3.2, 3.7, 3.2\}$
- Algorithm 4.2.2 generates the linked list:



- Node order is determined by data order in the sample
- Alternatives:
  - Use `count` to maintain the list in order of decreasing frequency
  - Use `value` to sort the list by data value

Program ddh

- Generates a discrete-data histogram, based on algorithm 4.2.2 and the linked-list data structure
- Valid for integer-valued and real-valued input
- No outlier checks
- No restriction on sample size
- Supports file redirection
- Assumes $|\mathcal{X}|$ is small, a few hundred or less
- Requires $\mathcal{O}(|\mathcal{X}|)$ computation *per sample value*
- Should *not* be used on continuous samples (see Section 4.3)

Example 4.2.5

Construct a histogram of the inventory level (prior to review) for the simple inventory system

- Remove summary statistics from `sis2`
- Print the inventory within the `while` loop in `main`:

Printing Inventory

```
⋮
index++;
printf( ``% ld \ n'', inventory); /* This line is new */
if (inventory < MINIMUM) {
⋮
```

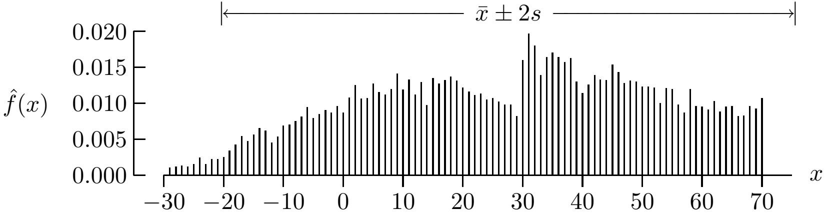
- If the new executable is `sis2mod`, the command

  sis2mod | ddh > sis2.out

  will produce a discrete-data histogram file `sis2.out`

Example 4.2.6

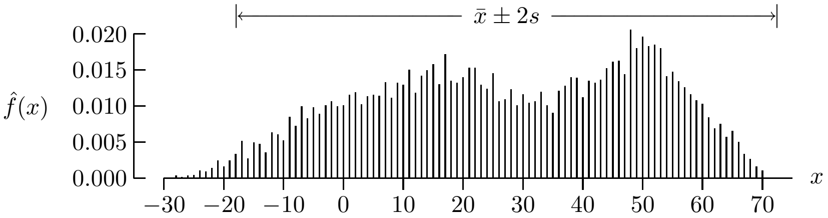
Using `sis2mod` to generate 10 000 weeks of sample data, the inventory level histogram from `sis2.out` can be constructed



- x denotes the inventory level prior to review
- $\bar{x} = 27.63$ and $s = 23.98$
- About 98.5% of the data falls within $\bar{x} \pm 2s$

Example 4.2.7

Change the demand to *Equilikely*(5, 25)+*Equilikely*(5, 25) in `sis2mod` and construct the new inventory level histogram

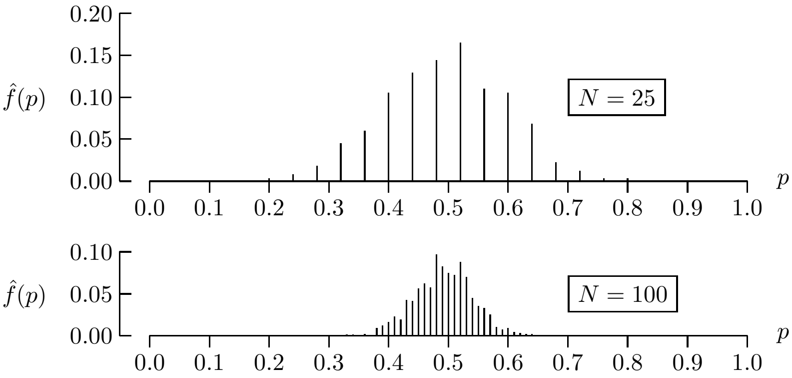


- More tapered at the extreme values
- $\bar{x} = 27.29$ and $s = 22.59$
- About 98.5% of the data falls within $\bar{x} \pm 2s$

Example 4.2.8

- Monte Carlo simulation was used to generate 1000 point estimates of the probability of winning the dice game *craps*
- Sample is $\mathcal{S} = \{p_1, p_2, \ldots, p_{1000}\}$ with $p_i = \frac{\text{\# wins in } N \text{ plays}}{N}$
- $\mathcal{X} = \{0/N, 1/N, \ldots, N/N\}$
- Compare $N = 25$ and $N = 100$ plays per estimate
- $\mathcal{X}$ is small, can use discrete-data histograms

Histograms of Probability of Winning Craps



- $N = 25$: $\bar{p} = 0.494$ $s = 0.102$
- $N = 100$: $\bar{p} = 0.492$ $s = 0.048$
- *Four-fold* increase in number of replications produces *two-fold* reduction in uncertainty

Empirical Cumulative Distribution Functions

- Histograms estimate distributions
- Sometimes, the *cumulative* version is preferred:

$$\hat{F}(x) = \frac{\text{the number of } x_i \in \mathcal{S} \text{ with } x_i \leq x}{n}$$

- Useful for quantiles

Example 4.2.9

The empirical cumulative distribution function for $N = 100$ from Example 4.2.8 (in four different styles):