Discrete-Event Simulation:

A First Course

Section 6.1: Discrete Random Variables

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Section 6.1: Discrete Random Variables

- A random variable X is *discrete* if and only if its set of possible values $\mathcal{X}$ is finite or, at most, countably infinite
- A discrete random variable X is uniquely determined by
  - Its set of possible values $\mathcal{X}$
  - Its *probability density function* (pdf):
    A real-valued function $f(\cdot)$ defined for each $x \in \mathcal{X}$ as the probability that X has the value x

$$f(x) = Pr(X = x)$$

By definition,

$$\sum_x f(x) = 1$$

Examples

- **Example 6.1.1** X is *Equilikely*(a, b)

  $|\mathcal{X}| = b - a + 1$ and each possible value is equally likely

$$f(x) = \frac{1}{b - a + 1} \qquad x = a, a + 1, \ldots, b$$

- **Example 6.1.2** Roll two fair face

  If X is the sum of the two up faces, $\mathcal{X} = \{x | x = 2, 3, \ldots, 12\}$

  From example 2.3.1,

$$f(x) = \frac{6 - |7 - x|}{36} \qquad x = 2, 3, \ldots, 12$$

Example 6.1.3

- A coin has p as its probability of a head
- Toss it until the *first* tail occurs
- If X is the number of heads, $\mathcal{X} = \{x | x = 0, 1, 2, \ldots\}$ and the pdf is

$$f(x) = p^x(1-p) \qquad x = 0, 1, 2, \ldots$$

- X is *Geometric*(p) and the set of possible values is *infinite*
- Verify that $\sum_x f(x) = 1$:

$$\sum_x f(x) = \sum_{x=0}^{\infty} p^x(1-p) = (1-p)(1+p+p^2+p^3+p^4+\cdots) = 1$$

Cumulative Distribution Function

- The *cumulative distribution function(cdf)* of the discrete random variable X is the real-valued function $F(\cdot)$ for each $x \in \mathcal{X}$ as

$$F(x) = Pr(X \leq x) = \sum_{t \leq x} f(t)$$

- If X is *Equilikely*(a, b) then the cdf is

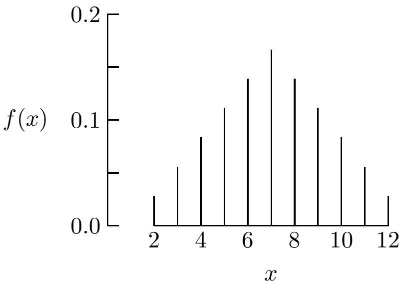
$$F(x) = \sum_{t=a}^{x} 1/(b-a+1) = (x-a+1)/(b-a+1) \qquad x = a, a+1, \ldots, b$$

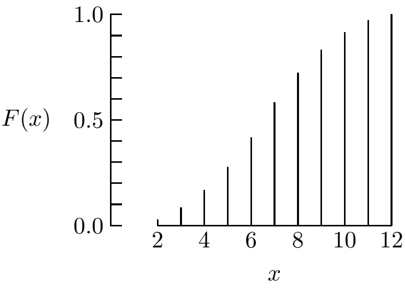
- If X is *Geometric*(p) then the cdf is

$$F(x) = \sum_{t=0}^{x} p^t(1-p) = (1-p)(1+p+\cdots+p^x) = 1-p^{x+1} \qquad x = 0, 1, 2, \ldots$$

Example 6.1.5

- No simple equation for $F(\cdot)$ for sum of two dice
- $|\mathcal{X}|$ is small enough to tabulate the cdf





Relationship Between cdfs and pdfs

- A cdf can be generated from its corresponding pdf by recursion

  For example, $\mathcal{X} = \{x|x = a, a+1, ..., b\}$

$$\begin{aligned} F(a) &= f(a) \\ F(x) &= F(x-1) + f(x) \qquad x = a+1, a+2, ..., b \end{aligned}$$

- A pdf can be generated from its corresponding cdf by subtraction

$$\begin{aligned} f(a) &= F(a) \\ f(x) &= F(x) - F(x-1) \qquad x = a+1, a+2, ..., b \end{aligned}$$

- A discrete random variable can be defined by specifying *either* its pdf or its cdf

Other cdf Properties

- A cdf is strictly monotone increasing:
  if $x_1 < x_2$, then $F(x_1) < F(x_2)$
- The cdf values are bounded between 0.0 and 1.0
- Monotonicity of $F(\cdot)$ is the basis to generate discrete random variates in the next section

Mean and Standard Deviation

- The *mean* μ of the discrete random variable X is

$$\mu = \sum_x x f(x)$$

- The corresponding *standard deviation* σ is

$$\sigma = \sqrt{\sum_x (x-\mu)^2 f(x)} \qquad \text{or} \qquad \sigma = \sqrt{\left(\sum_x x^2 f(x)\right) - \mu^2}$$

- The *variance* is σ^2

Examples

- If X is *Equilikely*(a, b) then the mean and standard deviation are

$$\mu = \frac{a+b}{2} \qquad \text{and} \qquad \sigma = \sqrt{\frac{(b-a+1)^2 - 1}{12}}$$

When X is *Equilikely*$(1, 6)$, $\mu = 3.5$ and $\sigma = \sqrt{\frac{35}{12}} \cong 1.708$

- If X is the sum of two dice then

$$\mu = \sum_{x=2}^{12} x f(x) = 7 \quad \text{and} \quad \sigma = \sqrt{\sum_{x=2}^{12} (x-\mu)^2 f(x)} = \sqrt{35/6} \cong 2.415$$

Another Example

- If X is *Geometric*(p) then the mean and standard deviation are

$$\mu = \sum_{x=0}^{\infty} x f(x) = \sum_{x=1}^{\infty} x p^x (1-p) = \cdots = \frac{p}{1-p}$$

$$\sigma^2 = \left(\sum_{x=0}^{\infty} x^2 f(x)\right) - \mu^2 = \left(\sum_{x=1}^{\infty} x^2 p^x (1-p)\right) - \frac{p^2}{(1-p)^2}$$

$$\vdots$$

$$\sigma^2 = \frac{p}{(1-p)^2}$$

$$\sigma = \frac{\sqrt{p}}{(1-p)}$$

Expected Value

- The mean of a random variable is also known as the *expected value*
- The expected value of the discrete random variable X is

$$E[X] = \sum_x x f(x) = \mu$$

- Expected value refers to the *expected average* of a large sample $x_1, x_2, \ldots, x_n$ corresponding to X: $\bar{x} \to E[X] = \mu$ as $n \to \infty$.
- The *most likely* value x (with largest $f(x)$) is the *mode*, which can be different from the expected value

Example 6.1.10

- Toss a fair coin until the first tail appears
- The most likely number of heads is 0
- The expected number of heads is 1
- 0 occurs with probability $1/2$ and 1 occurs with probability $1/4$

  The most likely value is twice as likely as the expected value
- For some random variables, the mean and mode may be the same

  For the sum of two dice, the most likely value and expected value are both 7

More on Expectation

- Define function $h(\cdot)$ for all possible values of X
  $h(\cdot) : \mathcal{X} \to \mathcal{Y}$
- $Y = h(X)$ is a *new* random variable, with possible values $\mathcal{Y}$
- The expected value of Y is

$$E[Y] = E[h(X)] = \sum_x h(x)f(x)$$

Note: in general, this is *not* equal to $h(E[X])$

Example 6.1.11

- If $y = (x-\mu)^2$ with $\mu = E[X]$,

$$E[Y] = E[(X-\mu)^2] = \sum_x (x-\mu)^2 f(x) = \sigma^2$$

- If $y = x^2 - \mu^2$,

$$E[Y] = E[X^2 - \mu^2] = \sum_x (x^2 - \mu^2) f(x) = \left(\sum_x x^2 f(x)\right) - \mu^2 = \sigma^2$$

- So that $\sigma^2 = E[X^2] - E[X]^2$
- $E[X^2] \geq E[X]^2$ with equality if and only if X is not really random

Example 6.1.12

- If $Y = aX + b$ for constants a and b,

$$E[Y] = E[aX+b] = \sum_x (ax+b)f(x) = a\left(\sum_x xf(x)\right)+b = aE[X]+b$$

- Suppose
  - X is the number of heads before the first tail
  - Win \$2 for every head and let Y be the amount you win
- The possible values Y you win are defined by

$$y = h(x) = 2x \qquad x = 0, 1, 2, \ldots$$

- Your *expected winnings* are

$$E[Y] = E[2X] = 2E[X] = 2$$

Discrete Random Variable Models

- A random *variable* is an abstract, but well defined, mathematical object
- A random *variate* is an algorithmically generated possible value of a random *variable*
- For example, the functions `Equilikely` and `Geometric` generate random *variates* corresponding to *Equilikely*(a, b) and *Geometric*(p) random *variables*, respectively

Bernoulli Random Variable

- The discrete random *variable* X with possible values $\mathcal{X} = \{0, 1\}$
- $X = 1$ with probability p and $X = 0$ with probability $1 - p$
- The pdf: $f(x) = p^x(1-p)^{1-x} \quad \text{for} \quad x \in \mathcal{X}$
- The cdf: $F(x) = (1-p)^{1-x} \quad \text{for} \quad x \in \mathcal{X}$
- The mean: $\mu = 0 \cdot (1-p) + 1 \cdot p = p$
- The variance: $\sigma^2 = (0-p)^2(1-p) + (1-p)^2 p = p(1-p)$
- The standard deviation: $\sigma = \sqrt{p(1-p)}$

Bernoulli Random Variate

- To generate a *Bernoulli*(p) random *variate*

Generating a Bernoulli Random Variate

```
if (Random()< 1.0-p)
    return 0;
else
    return 1;
```

- Monte Carlo simulation that uses n replications to estimate an unknown probability p is equivalent to generating an *iid* sequence of n *Bernoulli*(p) random variates

Example 6.1.14

- *Pick-3* Lottery: pick a 3-digit number between 000 and 999
- Costs \$1 to play the game and wins \$500 if a player matches the 3-digit number chosen by the state
- Let $Y = h(X)$ be the player's yield

$$h(x) = \begin{cases} -1 & \text{x=0} \\ 499 & \text{x=1} \end{cases}$$

- The player's *expected* yield is

$$E[Y] = \sum_{0}^{1} h(x)f(x) = h(0)(1-p) + h(1)p = \cdots = -0.5$$

Binomial Random Variable

- A coin has p as its probability of a head and toss this coin n times
- Let X be the number of heads; X is a $\mathit{Binomial}(n, p)$ random variable
- $\mathcal{X} = \{0, 1, 2, \cdots, n\}$ and the pdf is

$$f(x) = \binom{n}{x} p^x (1 - p)^{n-x} \qquad x = 0, 1, 2, \cdots, n$$

- n tosses of the coin generate an *iid* sequence $X_1, X_2, \cdots, X_n$ of $\mathit{Bernoulli}(p)$ random variables and $X = X_1 + X_2 + \cdots + X_n$

Verify that $\sum_x f(x) = 1$

- *Binomial equation*

$$(a+b)^n = \sum_{x=0}^{n} \binom{n}{x} a^x b^{n-x}$$

- In the particular case where $a = p$ and $b = 1 - p$

$$1 = (1)^n = (p + (1-p))^n = \sum_{x=0}^{n} \binom{n}{x} p^x (1-p)^{n-x}$$

Mean and Variance of *Binomial*(n, p)

- The mean is

$$\mu = E[X] = \sum_{x=0}^{n} x f(x) = \sum_{x=0}^{n} x \binom{n}{x} p^x (1-p)^{n-x}$$

$$= np \sum_{x=1}^{n} \frac{(n-1)!}{(x-1)!(n-x)!} p^{x-1} (1-p)^{n-x}$$

Let $m = n - 1$ and $t = x - 1$

$$\mu = np \sum_{t=0}^{m} \frac{m!}{t!(m-t)!} p^t (1-p)^{m-t} = np(p+(1-p))^m = np(1)^m = np$$

- The variance is

$$\sigma^2 = E[X^2] - \mu^2 = \left(\sum_{x=0}^{n} x^2 f(x) \right) - \mu^2 = \cdots = np(1-p)$$

Pascal Random Variable

- A coin has p as its probability of a head and toss this coin until the n^{th} tail occurs
- If X is the number of heads, X is a *Pascal*(n, p) random variable
- $\mathcal{X}$={0,1,2,...} and the pdf is

$$f(x) = \binom{n+x-1}{x} p^x (1-p)^n \qquad x = 0, 1, 2, \dots$$

Pascal Random Variable ctd.

- Negative binomial expansion:

$$(1-p)^{-n} = 1+\binom{n}{1}p+\binom{n+1}{2}p^2+\cdots+\binom{n+x-1}{x}p^x+\cdots$$

- Prove that the infinite pdf sum converges to 1

$$\sum_{x=0}^{\infty}\binom{n+x-1}{x}p^x(1-p)^n = (1-p)^n(1-p)^{-n} = 1$$

- It can also be shown that

$$\mu = E[X] = \sum_{x=0}^{\infty} xf(x) = \cdots = \frac{np}{1-p}$$

$$\sigma^2 = E[X^2]-\mu^2 = \left(\sum_{x=0}^{\infty} x^2 f(x)\right) - \mu^2 = \cdots = \frac{np}{(1-p)^2}$$

Example 6.1.17

- If $n > 1$ and $X_1, X_2, \ldots, X_n$ is an *iid* sequence of n *Geometric*(p) random variables, the sum is a *Pascal*(n, p) random variable
- For example,if $n = 4$ and p is large, a head/tail sequence might be

$$\underbrace{hhhhhht}_{X_1=6}\ \underbrace{hhhhhhhhht}_{X_2=9}\ \underbrace{hhhht}_{X_3=4}\ \underbrace{hhhhhhht}_{X_4=7}$$

$$X = X_1 + X_2 + X_3 + X_4 = 26$$

- We see that a *Pascal*(n, p) random variable is the sum of *iid* *Geometric*(p) random variables

Poisson Random Variable

- *Poisson*(μ) is a limiting case of *Binomial*$(n, \mu/n)$
- Fix μ and x as $n \to \infty$

$$f(x) = \frac{n!}{x!(n-x)!} \left(\frac{\mu}{n}\right)^x \left(1 - \frac{\mu}{n}\right)^{n-x} = \frac{\mu^x}{x!} \left(\frac{n!}{(n-x)!(n-\mu)^x}\right) \left(1 - \frac{\mu}{n}\right)^n$$

It can be shown that

$$\lim_{n\to\infty} \left(\frac{n!}{(n-x)!(n-\mu)^x}\right) = 1 \qquad \text{and} \qquad \lim_{n\to\infty} \left(1 - \frac{\mu}{n}\right)^n = exp(-\mu)$$

So that

$$\lim_{n\to\infty} f(x) = \frac{\mu^x}{x!} exp(-\mu)$$