Discrete-Event Simulation:

A First Course

Section 7.2: Generating Continuous Random Variates

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Section 7.2: Generating Continuous Random Variates

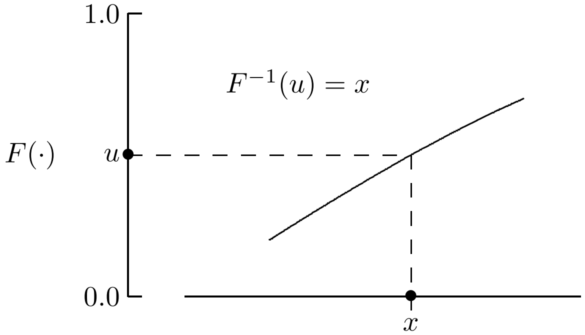
- The *inverse distribution function (idf)* of X is the function $F^{-1} : (0,1) \rightarrow \mathcal{X}$ for all $u \in (0,1)$ as

$$F^{-1}(u) = x$$

  where $x \in \mathcal{X}$ is the unique possible value for $F(x) = u$
- There is a one-to-one correspondence between possible values $x \in \mathcal{X}$ and cdf values $u = F(x) \in (0,1)$
  - Assumes the cdf is strictly monotone increasing
  - True if $f(x) > 0$ for all $x \in \mathcal{X}$

Continuous Random Variable idfs

- Unlike the a discrete random variable, the idf for a continuous random variable is a true inverse



- Can sometimes determine the idf in "closed form" by solving $F(x) = u$ for x

Examples

- If X is *Uniform*(a, b), $F(x) = (x - a)/(b - a)$ for $a < x < b$

$$x = F^{-1}(u) = a + (b - a)u \qquad 0 < u < 1$$

- If X is *Exponential*(μ), $F(x) = 1 - \exp(-x/\mu)$ for $x > 0$

$$x = F^{-1}(u) = -\mu \ln(1 - u) \qquad 0 < u < 1$$

- If X is a continuous variable with possible value $0 < x < b$ and pdf $f(x) = 2x/b^2$, the cdf is $F(x) = (x/b)^2$

$$x = F^{-1}(u) = b\sqrt{u} \qquad 0 < u < 1$$

Random Variate Generation By Inversion

- X is a continuous random variable with idf $F^{-1}(\cdot)$
- Continuous random variable U is *Uniform*$(0, 1)$
- Z is the continuous random variable defined by $Z = F^{-1}(U)$

Theorem (7.2.1)

Z and X are identically distributed

Algorithm 7.2.1

If X is a continuous random variable with idf $F^{-1}(\cdot)$, a continuous random variate x can be generated as

```
u = Random();
return F^{-1}(u);
```

Inversion Examples

Example 7.2.4: Generating a *Uniform*(a, b) Random Variate

```
u = Random();
return a + (b - a) * u;
```

Example 7.2.5: Generating an *Exponential*(μ) Random Variate

```
u = Random();
return −μ * log(1 - u);
```

Note: `return −μ * log(1 - u)` is prefered to `return −μ * log(u)`, though both generate an Exponential random variate

Examples 7.2.4 and 7.2.5

- Algorithms in Example 7.2.4 and 7.2.5 are ideal
- Both are portable, exact, robust, efficient, clear, synchronized and monotone
- It is not always possible to solve for a continuous random variable idf explicitly by algebraic techniques
- Two other options may be available
  - Use a function that accurately *approximates* $F^{-1}(\cdot)$
  - Determine the idf by solving $u = F(x)$ *numerically*

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Approximate Inversion

- If Z is a *Normal*$(0, 1)$, the cdf is the special function $\Phi(\cdot)$
- The idf $\Phi^{-1}(\cdot)$ cannot be evaluated in closed form
- The idf can be *approximated* as the ratio of two fourth degree polynomials (Odeh and Evans, 1974)
- The approximation is efficient and essentially has negligible error

Approximation of $\Phi(\cdot)$

- For any $u \in (0, 1)$, a *Normal*$(0, 1)$ idf approximation is $\Phi^{-1}(u) \simeq \Phi_a^{-1}(u)$ where

$$\Phi_a^{-1}(u) = \begin{cases} -t + p(t)/q(t) & 0.0 < u < 0.5 \\ t - p(t)/q(t) & 0.5 \le u < 1.0 \end{cases}$$

  and

$$t = \begin{cases} \sqrt{-2\ln(u)} & 0.0 < u < 0.5 \\ \sqrt{-2\ln(1-u)} & 0.5 \le u < 1.0 \end{cases}$$

  and

$$p(t) = a_0 + a_1 t + \cdots + a_4 t^4$$

$$q(t) = b_0 + b_1 t + \cdots + b_4 t^4$$

- The ten coefficients can be chosen to produce an absolute error less than 10^{-9} for all $0.0 < u < 1.0$

Example 7.2.6

- Inversion can be used to generate *Normal*(0, 1) variates:

Example: 7.2.6: Generating a *Normal*(0, 1) Random Variate

```
u = Random();
return Φ_a^{-1}(u);
```

$$\texttt{return}\ \Phi_a^{-1}(u);$$

- This algorithm is portable, essentially exact, robust, reasonably efficient, synchronized and monotone
- Clarity?

Alternative Method 1

- If $U_1, U_2, \ldots, U_{12}$ is an *iid* sequence of *Uniform*$(0,1)$,

$$Z = U_1 + U_2 + \ldots + U_{12} - 6$$

  is approximately *Normal*$(0,1)$
  - The mean is 0.0 and the standard deviation is 1.0
  - Possible values are $-6.0 < z < 6.0$
  - Justification is provided by the central limit theorem (Section 8.1)
  - This algorithm is: portable, robust, relatively efficient and clear
  - This algorithm is **not**: exact, synchronized or monotone

Alternative Method 2

- If U_1 and U_2 are independent *Uniform*$(0, 1)$ RVs then

$$Z_1 = \sqrt{-2\ln(U_1)}\cos(2\pi U_2)$$

and

$$Z_2 = \sqrt{-2\ln(U_1)}\sin(2\pi U_2)$$

will be independent *Normal*$(0, 1)$ RVs (Box and Muller, 1958)

- This algorithm is: portable, exact, robust and relatively efficient;
- This algorithm is **not**: clear or monotone
- The algorithm is synchronized only in pair-wise fashion

Normal and Lognormal Random Variates

- Random variates corresponding to *Normal*(μ, σ) and *Lognormal*(a, b) can be generated by using a *Normal*$(0, 1)$ random variate generator

Example 7.2.7: Generating a *Normal*(μ, σ) Random Variate

```
z = Normal(0.0, 1.0);
return μ + σ * z;
/* see Definition 7.1.7 */
```

Example 7.2.8: Generating a *Lognormal*(a, b) Random Variate

```
z = Normal(0.0, 1.0);
return exp(a + b * z);
/* see Definition 7.1.8 */
```

- Both algorithms are essentially ideal

Numerical Inversion

- *Numerical* inversion provides another way to generate continuous random variates; that is, $u = F(x)$ can be solved for x iteratively
- *Newton's method* provides a good compromise between rate of convergence and robustness
- Given $u \in (0, 1)$, let t be close to the value of x for which $u = F(x)$
- If $F(\cdot)$ is expanded in a Taylor's series about the point t

$$F(x) = F(t) + F'(t)(x - t) + \frac{1}{2!}F''(t)(x - t)^2 + \cdots$$

- Recall $F'(t) = f(t)$
- For small $|x - t|$, ignore $(x - t)^2$ and higher order terms

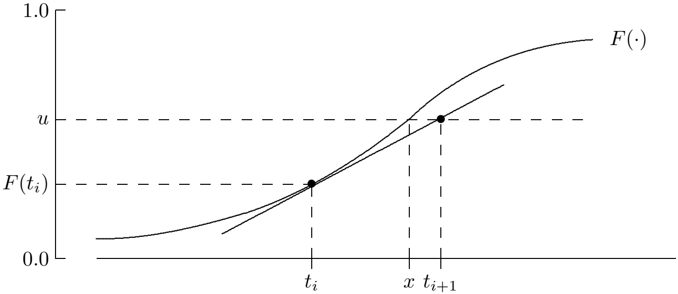
Newton's Method

- Set $u = F(x) \simeq F(t) + f(t)(x - t)$ and solve for x to obtain

$$x \simeq t + \frac{u - F(t)}{f(t)}$$

- Use initial guess t_0 and iterate to solve for x: $t_i \to x$ as $i \to \infty$

$$t_{i+1} = t_i + \frac{u - F(t_i)}{f(t_i)} \qquad i = 0, 1, 2, \cdots$$



Two Issues Relative to Newton's Method

- The choice of an initial value t_0
  - The best choice for the initial value is the mode
  - For most continuous RVs described in text, $t_0 = \mu$ is an essentially equivalent choice
- The test for convergence
  - Given a convergence parameter $\epsilon > 0$
  - Iterate until $|t_{i+1} - t_i| < \epsilon$

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Algorithm 7.2.2

Algorithm 7.2.2

Given $u \in (0, 1)$, the pdf $f(\cdot)$, the cdf $F(\cdot)$ and a convergence parameter $\epsilon > 0$, this algorithm will solve for $x = F^{-1}(u)$

```
x = μ; /*μ is E[X]*/
do {
    t = x;
    x = t + (u - F(t)) / f(t);
} while (|x-t| > ϵ);
return x; /* x is F^{-1}(u)*/
```

- If u is small and X is non-negative, a negative value of x may occur early in the iterative process.
- Negative t will cause $F(t)$ and $f(t)$ to be undefined for positive RVs

Modified Algorithm 7.2.2

- The following modification can be used to avoid the problem

Modified Algorithm 7.2.2

```
x = μ; /*μ is E[X]*/
do {
  t = x;
  x = t + (u - F(t)) / f(t);
  if (x <= 0.0)
    x = 0.5 * t;
} while (|x-t| > ε);
return x; /* x is F^{-1}(u)*/
```

- Algorithms 7.2.1 and 7.2.2 together provide a general purpose inversion approach to continuous random variate generation
- E.g., the *Erlang*(n, b) idf function in `rvms` is based on Alg.7.2.2 and can be used with Algorithm 7.2.1

Alternative Random Variate Generation Algorithms

- **Erlang Random Variates**

  An *Erlang*(n, b) random variate can be generated by summing n *Exponential*(b) random variates

Generating an *Erlang*(n, b) Random Variate

```
x = 0.0;
for (i = 0; i < n; i++)
    x += Exponential(b);
return x;
```

- The algorithm is: portable, exact, robust, and clear
- The algorithm is **not** efficient (it is $\mathcal{O}(n)$), synchronized or monotone

Modified Algorithms for Erlang Random Variates

- To increase computational efficiency, use

Generating an *Erlang*(n, b) Random Variate

```
t = 1.0;
for (i = 0; i < n; i++)
  t *= (1.0 - Random());
return -b * log(t);
```

- This algorithm requires only one `log()` evaluation, rather than n
- Can further improve efficiency by using `t *= Random();`
- The algorithm remains $\mathcal{O}(n)$, so is not efficient if n is large

Chisquare Random Variates

- If n is an even positive integer, an *Erlang*$(n/2, 2)$ random variate is equivalent to a *Chisquare*(n) random variable
- X is a *Chisquare*(n) random variable iff $X = Z_1^2 + Z_2^2 + \cdots + Z_n^2$ where $Z_1, Z_2, \ldots, Z_n$ are *iid* *Normal*$(0, 1)$ random variables

Generating a *Chisquare*(n) Random Variate

```
x = 0.0;
for (i = 0; i < n; i++){
    z = Normal(0.0,
1.0);
    x += (z * z); }
return x;
```

- The algorithm is: portable, exact, robust, clear
- The algorithm is **not**: efficient(it is $\mathcal{O}(n)$), synchronized or monotone

Student Random Variates

- X is *Student*(n) iff $X = Z/\sqrt{V/n}$ where
  - Z is *Normal*$(0, 1)$
  - V is *Chisquare*(n)
  - Z and V are independent

Generating a *Student*(n) Random Variate

```
z = Normal(0.0, 1.0);
v = Chisquare(n);
return z / sqrt(v / n);
```

- The algorithm is: portable, exact, robust, clear
- The algorithm is **not** synchronized or monotone
- Efficiency depends on algs. used for *Normal* and *Chisquare*

Testing for Correctness using Histograms

- A natural way to do this at the computational level is:
  - use the algorithm to generate a sample of n random variates and construct a k-bin continuous-data histogram with bin width δ
  - $\hat{f}$ is the histogram density and $f(x)$ is the pdf

$$\hat{f} \to f(x) \qquad \text{as} \qquad n \to \infty \qquad \text{and} \qquad \delta \to 0$$

- In practice, using a large but finite value of n and a small but non-zero value of δ, perfect agreement between $\hat{f}(x)$ and $f(x)$ will not be achieved
  - In the discrete case, it is due to natural sampling variability
  - In the continuous case, the *quantization error* associated with binning the sample is an additional factor

Quantization Error

- Let $\mathcal{B} = [m - \delta/2, m + \delta/2]$ be a small histogram bin
- Use the Taylor expansion of $f(x)$ at $x = m$

$$f(x) = f(m) + f'(m)(x-m) + \frac{1}{2!}f''(m)(x-m)^2 + \frac{1}{3!}f'''(m)(x-m)^3 + \cdots$$

- The probability of falling within the bin is

$$\Pr(x \in \mathcal{B}) = \int_{\mathcal{B}} f(x)dx = \cdots = f(m)\delta + \frac{1}{24}f''(m)\delta^3 + \cdots$$

Quantization Error (2)

- For all $x \in \mathcal{B}$, the histogram density is

$$\hat{f}(x) = \frac{1}{\delta} \Pr(X \in \mathcal{B}) \simeq f(m) + \frac{1}{24} f''(m)\delta^2$$

- Unless $f''(m) = 0$, there is a positive or negative *bias* between
  - $\hat{f}(x)$, the experimental density of the histogram bin and
  - $f(m)$, the theoretical pdf evaluated at the bin midpoint
- This bias may be significant if the curvature of the pdf is large at the bin midpoint

Example 7.2.9

- X is a continuous random variable with pdf

$$f(x) = \frac{2}{(x+1)^3} \qquad x > 0$$

- The cdf X is

$$F(x) = \int_0^x f(t)dt = 1 - \frac{1}{(x+1)^2} \quad x > 0$$

- The idf is

$$F^{-1}(u) = \frac{1}{\sqrt{1-u}} - 1 \qquad 0 < u < 1$$

- Note the pdf curvature is very large close to $x = 0$; therefore, the histogram will not match the pdf well for the bins close to $x = 0$

Example 7.2.9 ctd.

- Random variates for X can be generated using inversion
- Correctness of the inversion can be tested by constructing a histogram
- Using histogram bin widths of $\delta = 0.5$, as $n \to \infty$, $\hat{f}(x)$ and $f(m)$ are (with *d.dddd* precision):

m	:	0.25	0.75	1.25	1.75	2.25	2.75	$\cdots$
$\hat{f}(x)$	:	1.1111	0.3889	0.1800	0.0978	0.0590	0.0383	
$f(m)$	:	1.0240	0.3732	0.1756	0.0962	0.0583	0.0379	

- For the first bin ($m = 0.25$), the curvature bias is

$$\frac{1}{24} f''(m)\delta^2 = 0.08192$$

Testing for Correctness using the Empirical cdf

- Compare the empirical cdf (section 4.3) with the population cdf $F(x)$
- Eliminates binning quantization error
- For large samples (as $n \to \infty$), $\hat{F}(x) \to F(x)$

Library rvgs

- Contains 7 continuous random variate generators
  - double Chisquare(long n)
  - double Erlang(long n, double b)
  - double Exponential(double μ)
  - double Lognormal(double a, double b)
  - double Normal(double μ, double σ)
  - double Student(long n)
  - double Uniform(double a, double b)