Discrete-Event Simulation:

A First Course

Section 7.5: Nonstationary Poisson Processes

Section 7.5: Nonstationary Poisson Processes

- Suppose we want the arrival rate λ to change over time: $\lambda(t)$
- Recall the algorithm to generate a *stationary* Poisson process:

Stationary Poisson Process

```
$a_0$ = 0.0;
$n$ = 0;
while ($a_n$ < $\tau$) {
    $a_{n+1}$ = $a_n$ + Exponential(1 / $\lambda$);
    $n$++;
}
return $a_1$, $a_2$, ..., $a_{n-1}$;
```

- Above algorithm generates a stationary Poisson process
  - Time interval is $0 \leq t < \tau$
  - Event times are $a_1, a_2, a_3, \ldots$
  - Process has constant rate λ

Incorrect Algorithm

- Change constant λ to function:

Incorrect Algorithm

```
$a_0$ = 0.0;
$n$ = 0;
while ($a_n$ < $\tau$) {
    $a_{n+1}$ = $a_n$ + Exponential(1 / $\lambda(a_n)$);
    $n$++;
}
return $a_1$, $a_2$, ..., $a_{n-1}$;
```

- Incorrect: ignores future evolution of $\lambda(t)$ after $t = a_n$
- If $\lambda(a_n) < \lambda(a_{n+1})$ then $a_{n+1} - a_n$ will tend to be too large
- If $\lambda(a_n) > \lambda(a_{n+1})$ then $a_{n+1} - a_n$ will tend to be too small
- "Inertia error" will be small only if $\lambda(t)$ varies slowly with t

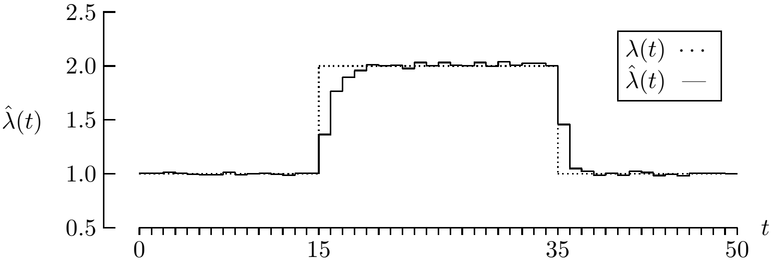
Example 7.5.1

Piecewise-constant rate function

$$\lambda(t) = \begin{cases} 1 & 0 \le t < 15 \\ 2 & 15 \le t < 35 \\ 1 & 35 \le t < 50 \end{cases}$$

- Simulated using incorrect algorithm for $\tau = 50$
- Process replicated 10000 times
- Partitioned interval $0 \le t \le 50$ into 50 bins
- Counted number of events for each bin, divided by 10000
- Result is $\hat{\lambda}(t)$, an *estimate* of $\lambda(t)$

Incorrect process generation



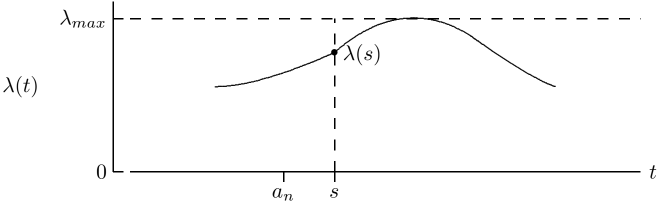
Incorrect algorithm has inertia error:

- $\hat{\lambda}(t)$ under-estimates $\lambda(t)$ after the rate increase
- $\hat{\lambda}(t)$ over-estimates $\lambda(t)$ after the rate decrease

Use one of 2 correct algorithms instead

Thinning method

- Due to Lewis and Shedler, 1979
- Uses an upper bound $\lambda_{\max} \geq \lambda(t)$ for $0 \leq t < \tau$
- Generates a stationary Poisson process with rate $\lambda_{\max}$
- Discards (thins) some events, probabilistically
  - Event at time s is kept with probability $\lambda(s)/\lambda_{\max}$



- Efficiency depends on $\lambda_{\max}$ being a tight bound

Algorithm 7.5.1

Algorithm 7.5.1

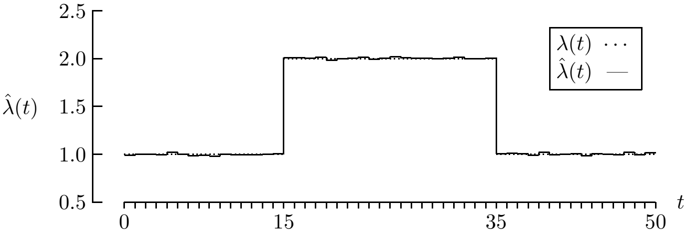
```
$a_0$ = 0.0;
$n$ = 0;
while ($a_n$ < $\tau$) {
    $s$ = $a_n$;
    do {
        $s$ = $s$ + Exponential(1 / $\lambda_{\max}$);
        $u$ = Uniform(0, $\lambda_{\max}$);
    } while ($u$ > $\lambda(s)$);
    $a_{n+1}$ = $s$;
    $n$++;
}
return $a_1$, $a_2$, ..., $a_{n-1}$;
```

When $\lambda(s)$ is low,

- The event at time s is more likely to be discarded
- The number of loop iterations is more likely to be large

Example 7.5.2

- The thinning method was applied to Example 7.5.1, using $\lambda_{\max} = 2$
- Computation time increased by a factor of about 2.2
- The algorithm is not synchronized
    - Even if a separate stream is used for `Uniform`



Inversion Method

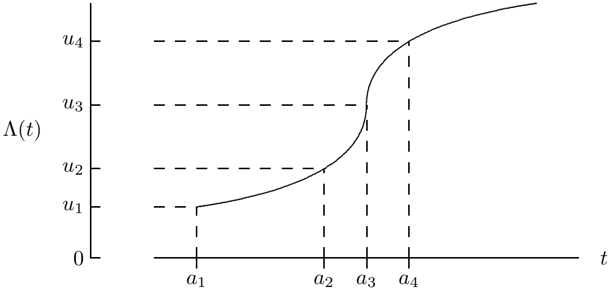
- Due to Çinlar, 1975
- Similar to inversion for random variate generation
- Requires only one call to `Random` per event
- Based upon the *cumulative* event rate function:

$$\Lambda(t) = \int_0^t \lambda(s)\, ds \qquad\qquad 0 \le t < \tau$$

- $\Lambda(t)$ represents the expected number of events in interval $[0, t)$
- If $\lambda(t) > 0$ then
  - $\Lambda(\cdot)$ is strictly monotone increasing
  - There exists an inverse $\Lambda^{-1}(\cdot)$

Algorithm 7.5.2: idea

- Generates a stationary "unit" Poisson process $u_1, u_2, u_3, \ldots$
  - Equivalent to n random points in interval $0 < u_i < \Lambda(\tau)$
- Each u_i is transformed into a_i using $\Lambda^{-1}(\cdot)$



Algorithm 7.5.2: details

Algorithm 7.5.2

```
$a_0$ = 0.0;
$u_0$ = 0.0;
$n$ = 0;
while ($a_n$ < $\tau$) {
    $u_{n+1}$ = $u_n$ + Exponential(1.0);
    $a_{n+1}$ = $\Lambda^{-1}(u_{n+1})$;
    $n$++;
} return $a_1$, $a_2$, ..., $a_{n-1}$;
```

- The algorithm is synchronized
- Useful when $\Lambda^{-1}(\cdot)$ can be evaluated efficiently

Example 7.5.3

- Use the rate function from Example 7.5.2

$$\lambda(t) = \begin{cases} 1 & 0 \le t < 15 \\ 2 & 15 \le t < 35 \\ 1 & 35 \le t < 50 \end{cases}$$

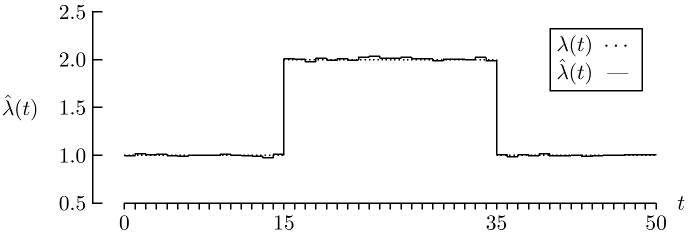
- By integration we obtain

$$\Lambda(t) = \begin{cases} t & 0 \le t < 15 \\ 2t - 15 & 15 \le t < 35 \\ t + 20 & 35 \le t < 50 \end{cases}$$

- Solving $u = \Lambda(t)$ for t we obtain

$$\Lambda^{-1}(u) = \begin{cases} u & 0 \le u < 15 \\ (u + 15)/2 & 15 \le u < 35 \\ u - 20 & 35 \le u < 50 \end{cases}$$

Example 7.5.3 results



- Generation time using inversion is similar to the incorrect algorithm
- For this example, $\Lambda(t)$ was easily inverted
- If $\Lambda(t)$ cannot be inverted in closed form
    - Use thinning if $\lambda_{\max}$ can be found
    - Use numerical methods to invert $\Lambda(t)$
    - Use an approximation

Next-Event Simulation Orientation

- The three algorithms must be adapted for next-event simulation:
  - Given current event time t, generate next event time

Incorrect Algorithm

```
arrival = $t$ + Exponential(1 / $\lambda(t)$);
```

Thinning Method

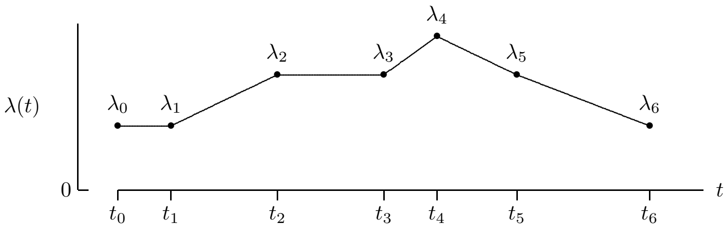
```
arrival = $t$;
do {
    arrival = arrival + Exponential(1 / $\lambda_{\max}$);
    $u$ = Uniform(0, $\lambda_{\max}$);
} while ($u$ > $\lambda$(arrival));
```

Inversion Method

```
arrival = $\Lambda^{-1}(\Lambda(t)$ + Exponential(1.0));
```

Piecewise-Linear Rate Functions

- A piecewise-constant rate function is (usually) unrealistic
- Obtaining an accurate estimate of $\lambda(t)$ is difficult
  - Requires lots of data — see Section 9.3
- We will examine piecewise-linear $\lambda(t)$ functions
  - Can be specified as a sequence of “knot pairs” (t_j, λ_j)



Algorithm 7.5.3

Algorithm 7.5.3 Step 1

- Given $k+1$ knot pairs (t_j, λ_j) with
    - $0 = t_0 < t_1 < \cdots < t_k = \tau$
    - $\lambda_j \geq 0$
- Four steps to construct
    - Piecewise-linear $\lambda(t)$
    - Piecewise-quadratic $\Lambda(t)$
    - $\Lambda^{-1}(u)$
- Define the slope of each segment

$$s_j = \frac{\lambda_{j+1} - \lambda_j}{t_{j+1} - t_j} \qquad j = 0, 1, \ldots, k-1$$

Algorithm 7.5.3 step 2/4

Algorithm 7.5.3 Step 2

- Define the *cumulative rate* for each knot point as

$$\Lambda_j = \int_0^{t_j} \lambda(t)\, dt \qquad\qquad j = 0, 1, \ldots, k$$

These can be computed recursively with

$$\begin{aligned} \Lambda_0 &= 0 \\ \Lambda_j &= \Lambda_{j-1} + \frac{1}{2}(\lambda_j + \lambda_{j-1})(t_j - t_{j-1}) \end{aligned}$$

Algorithm 7.5.3 step 3/4

Algorithm 7.5.3 Step 3

- For subinterval $t_j \leq t < t_{j+1}$

$$\begin{aligned} \lambda(t) &= \lambda_j + s_j(t - t_j) \\ \Lambda(t) &= \Lambda_j + \lambda_j(t - t_j) + \frac{1}{2} s_j (t - t_j)^2 \end{aligned}$$

If $s_j \neq 0$ then

  - $\lambda(t)$ is linear
  - $\Lambda(t)$ is quadratic

If $s_j = 0$ then

  - $\lambda(t)$ is constant
  - $\Lambda(t)$ is linear

Algorithm 7.5.3 step 4/4

Algorithm 7.5.3 Step 4

- For subinterval $\Lambda_j \le u < \Lambda_{j+1}$

$$\Lambda^{-1}(u) = t_j + \frac{2(u - \Lambda_j)}{\lambda_j + \sqrt{\lambda_j^2 + 2s_j(u - \Lambda_j)}}$$

If $s_j = 0$ then the above reduces to

$$\Lambda^{-1}(u) = t_j + \frac{(u - \Lambda_j)}{\lambda_j}$$

Algorithm 7.5.4: Inversion with piecewise-linear $\lambda(t)$

- Modified algorithm 7.5.2
- Also keeps track of index j, the current segment

Algorithm 7.5.4

```
$a_0$ = 0.0;
$u_0$ = 0.0;
$n$ = 0;
$j$ = 0;
while ($a_n$ < $\tau$) {
    $u_{n+1}$ = $u_n$ + Exponential(1.0);
    while (($\Lambda_{j+1}$ < $u_{n+1}$) and ($j$ < $k$))
        $j$++;
    $a_{n+1}$ = $\Lambda^{-1}(u_{n+1})$; /* $\Lambda_j < u_{n+1} \leq \Lambda_{j+1}$ */
    $n$++;
}
return $a_1$, $a_2$, ..., $a_{n-1}$;
```