Section 8.3: Finite-Horizon and Infinite-Horizon Statistics
Definition 8.3.1: Steady-state statistics

- Produced by simulating the operation of a stationary discrete-event system for an infinite length of time
- Steady-state statistics may not exist
Example 8.3.1

- Use \texttt{sis4} to simulate different numbers of time intervals, compute average inventory level $\bar{l}$ for each.
- Output from \texttt{sis4} for an increasing number of time intervals $n$:

  $n$: 20 40 80 160 320 640 1280 2560
  $\bar{l}$: 25.98 26.09 25.49 27.24 26.96 26.36 27.19 26.75

- As $n \to \infty$, $\bar{l}$ seems to converge to 26.75.
- Can only approximate $n \to \infty$ convergence.
- A better approach is to use interval estimation for the steady-state $\bar{l}$, using large but finite $n$.
- One way is to use \textit{batch means} — Section 8.4.
Example 8.3.2

- Use $ssq4$ to simulate different numbers of jobs $n$, compute $\bar{w}$, $\bar{l}$
- Use an arrival rate of $1/2$
- Change *Erlang* parameters so service rate is $1/2$ or less
- As $n$ increases, $\bar{w}$ and $\bar{l}$ will tend to increase, without limit
- Why? The server cannot keep up with demand, so the queue will grow as more jobs arrive
- In this case, the steady-state average wait and the average number in the node are both infinite
Steady-state statistics are known as *infinite-horizon* statistics.

An infinite-horizon discrete-event simulation is one for which the simulated operational time is effectively infinite.

A *finite-horizon* discrete-event simulation is one for which the simulated operational time is finite.

*Transient* system statistics are those statistics that are produced by a finite-horizon discrete-event simulation.

The initial conditions affect finite-horizon statistics.

The initial conditions *do not* affect infinite-horizon statistics: after enough time, the system loses memory of its initial state.
Example 8.3.3

- With minor modifications, program ssq4 can be used to simulate an initially idle $M/M/1$ service node processing a small number of jobs (say 100) and with relatively high traffic intensity (say 0.8).

- If the program is executed multiple times varying only the rngs initial seed from replication to replication,
  - the average wait in the node will vary significantly,
  - for most replications, the average wait will not be close to the steady-state average wait.

- If a relatively large number of jobs (say 10000) are used, variability of the average wait will be much less significant and be close to the steady-state average number.

- When the number of jobs becomes infinite, the initial condition bias disappears.
Another Important Distinction

- In an infinite-horizon simulation, the system “environment” is assumed to remain static.
  If the system is a single-server service node, both the arrival rate and the service rate are assumed to remain constant in time.
- In a finite-horizon simulation, no need to assume a static environment.
The “traditional” view: steady-state statistics are most important

- Steady-state statistics are better understood because they are much more easy to analyze mathematically
- It is frequently difficult to accurately model initial conditions and non-stationary system parameters

The “pragmatic” view: transient statistics are most important because steady-state is just a convenient fiction

- Depending on the application, both transient and steady-state statistics may be important
- Important to decide which statistics best characterize the system’s performance
Finite-horizon discrete-event simulations are also known as *terminating* simulations.

- In program ssq4, the system state is idle at the beginning and at the end of the simulation.
- The terminal condition is specified by the “close the door” time.
- The system state of sis4 is the current and on-order inventory levels; these states are the same at the beginning and at the end of the simulation.
- The terminal condition is specified by the number of time intervals.

Infinite-horizon discrete-event simulations must be terminated; typically done using whatever stopping conditions are most convenient.
The state variable $X(\cdot)$ is known formally as a stochastic process.

The typical objective of a finite-horizon simulation of this system would be to estimate the time-averaged transient statistic

$$\bar{X}(\tau) = \frac{1}{\tau} \int_{0}^{\tau} X(t) dt$$

where $\tau > 0$ is the terminal time.

The typical objective of an infinite-horizon simulation of this system would be to estimate the time-averaged steady-state statistics

$$\bar{x} = \lim_{\tau \to \infty} \bar{X}(\tau) = \lim_{\tau \to \infty} \frac{1}{\tau} \int_{0}^{\tau} X(t) dt$$

Note that $\bar{X}(\tau)$ is a random variable; $\bar{x}$ is not.
Definition 8.3.3

- If a discrete-event simulation is repeated, varying only the rngs initial states from run to run, each run of the simulation program is a replication and the totality of replications is said to be ensemble.

- Replications are used to generate estimates of the same transient statistic.

- The initial seed for each replication should be chosen to be no replication-to-replication overlap.

- The standard way is to use the final state of each rngs stream from one replication as the initial state for the next replication accomplished by calling PlantSeeds once outside the main replication loop.
Suppose the finite-horizon simulation is replicated \( n \) times, each time generating a state time history \( x_i(t) \)

\[
\bar{x}_i(\tau) = \frac{1}{\tau} \int_0^\tau x_i(t) dt
\]

where \( i = 1, 2, \ldots, n \) is the replication index

Each data point \( \bar{x}_i(\tau) \) is an independent observation of the random variable \( \bar{X}(\tau) \)

If \( n \) is large enough, the pdf of \( \bar{X}(\tau) \) can be estimated from a histogram of the \( \bar{x}_i(\tau) \).
Want $E[\bar{X}(\tau)]$ — point estimate is available as an *ensemble average*

$$\frac{1}{n} \sum_{i=1}^{n} \bar{x}_i(\tau)$$

An *interval* estimate for $E[\bar{X}(\tau)]$ can be calculated

- Use the interval estimation technique from Section 8.1
- Requires ensemble mean and standard deviation of $\bar{x}_i(\tau)$
Example 8.3.5

A modified version of sis4 was used to produce 21 replications

- 100 time intervals of operation were simulated
- Inventory parameters were \((s, S) = (20, 80)\).
- Measured \(\bar{L}(\tau)\), the time-averaged inventory level
- Each of these 21 numbers is a realization of \(\bar{L}(\tau)\)
Example 8.3.5, continued

- The mean and standard deviation are 27.06 and 1.76 respectively.
- If a 95% confidence interval estimate is desired,

\[ t^* = \text{idfStudent}(20, 0.975) = 2.086 \]

- The 95% confidence interval estimate is

\[ 27.06 \pm \frac{(2.086)(1.76)}{\sqrt{20}} = 27.06 \pm 0.82 \]

- We are 95% confident that \( E[\bar{L}(\tau)] \) is within interval 27.06 ± 0.82
Example 8.3.6

A modified version of ssq2 was used to produce 20 replications

- 100 jobs processed through $M/M/1$ service node
  - Node is initially idle
  - Arrival rate is $\lambda = 1.0$
  - Service rate is $\nu = 1.25$

- The resulting 20 observations of the average wait in the node:

  $1 \quad 3.33 \pm 1.04 \quad 12$

- From Algorithm 8.1.1, the resulting 95% confidence interval estimate is $3.33 \pm 1.04$
Example 8.3.6, continued

- Steady-state average wait for $M/M/1$ is

$$\frac{1}{\nu - \lambda}$$

(See Section 8.5)

- In our example
  - Arrival rate is $\lambda = 1.00$
  - Service rate is $\nu = 1.25$
  - Average wait is $\frac{1}{1.25 - 1.00} = 4.0$

- It is possible that 100 jobs are enough to produce steady-state statistics
Example 8.3.7

- The modified version of program ssq2 was used to produce 60 more replications.
- Consistent with $\sqrt{n}$ rule, expect two-fold decrease in the width of the interval estimate.
- Based on 80 replications, the resulting 95% confidence interval estimate was $3.25 \pm 0.39$.

In this case 100 jobs are not enough to produce steady-state statistics.
Example 8.3.8

- As a continuation of Example 8.3.6, the number of jobs per replication was increased from 100 to 1000.
- 20 replications were used to produce 20 observations of the average wait in the node.

Relative to Example 8.3.6, much more symmetric sample mean in Example 8.3.8.
Example 8.3.8, continued

- The 1000-jobs per replication results are more consistent with the underlying theory of interval estimation.
  - Requires a sample mean distribution that is approximately $\text{Normal}(\mu, \sigma/\sqrt{n})$.
  - Sample mean distribution is centered on (unknown) population mean.
- 1000 jobs may achieve steady-state; 100 jobs cannot.
- More on this issue in Section 8.4.