Discrete-Event Simulation:

A First Course

Section 8.4: Batch Means

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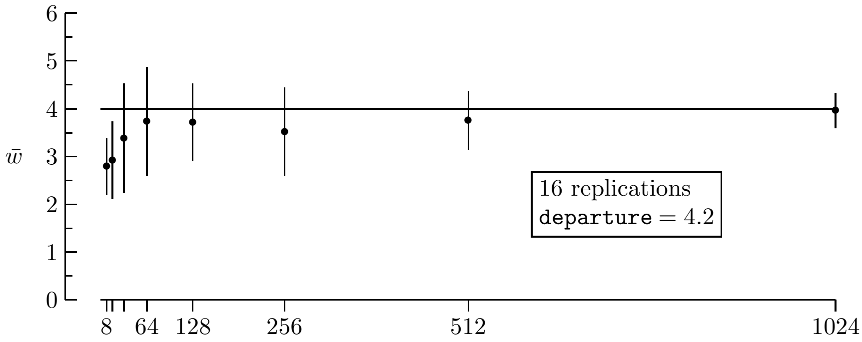
- Two types of DES models: transient and steady-state
- For transient, construct interval estimates using *replication*
- For steady-state, obtain *point* estimate by simulating for a long time
- Can we obtain *interval* estimates for steady-state statistics?

  $\longrightarrow$ *use method of batch means*

Example 8.4.1: Transient vs. Steady-State Estimates

- Program `ssq2` was modified to simulate an $M/M/1$ with $\lambda = 1.0$ and $\nu = 1.25$
- For an $M/M/1$, analytically:
  - Steady-state utilization is $\rho = \lambda/\nu = 0.8$ (Section 8.5)
  - Expected steady-state wait is $E[W] = 4$
- Can *transient* estimates be accurate *steady-state* estimates?
- Eliminate the initial state bias by setting `departure` to 4.2
- Use 16 replications to construct transient interval estimates for $8, 16, 32, \ldots, 1024$ jobs

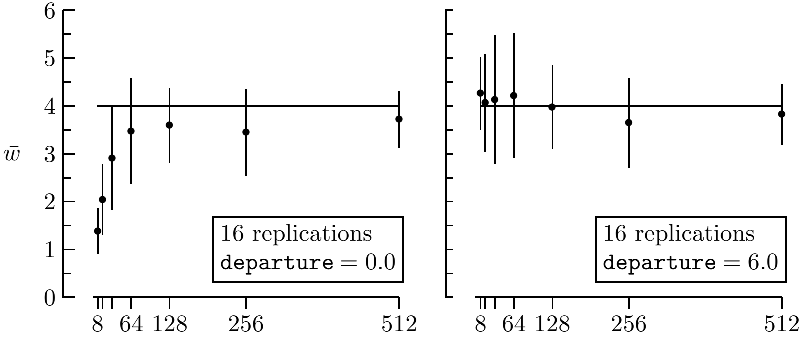
Example 8.4.1: Transient vs. Steady-State Estimates



- *Finite-horizon* interval estimates can be accurate *steady-state* estimates (provided the number of jobs is not small)

Example 8.4.2: Initial State Bias

- Consider initial values of 0.0 and 6.0 for `departure`
  - Most likely steady-state condition is an idle node (0.0)
  - The value 6.0 chosen by experimentation



Example 8.4.2: Observations

- `departure` = 0.0: initial state can have significant impact
- `departure` = 6.0: possible to choose an initial state that eliminates the initial state bias
- When the number of jobs is large, estimates are very good and essentially *independent* of the initial state

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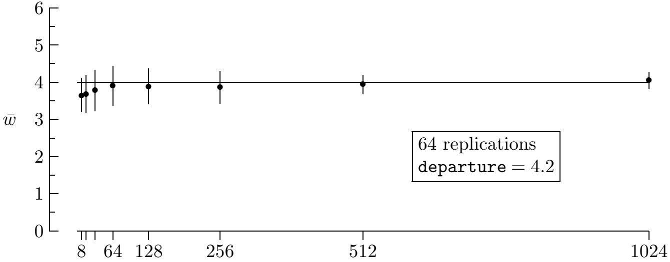
Interval Estimates for Steady-State

- Use replication-based *transient* interval estimates
- Each replication must correspond to a long simulated time period
- Three issues:
  - What is the initial state?
  - What is the length of the simulated time?
  - How many replications?
- Previous example provides insight into first two issues

Example 8.4.3: Increase the Number of Replications

- Repeat the previous experiments using 64 replications
- All other parameters remain the same



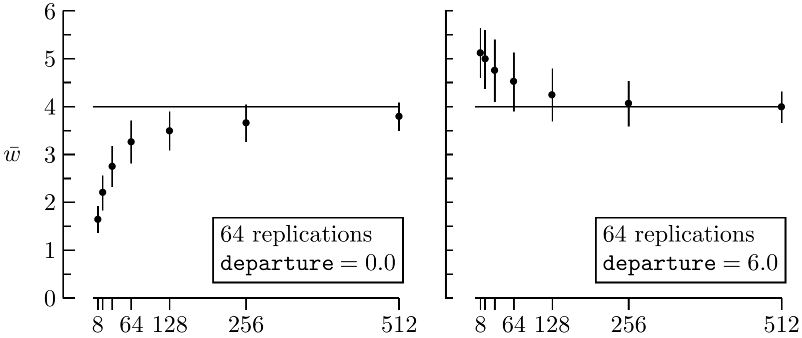
Example 8.4.3: Observations

- The interval estimates have been cut approximately in half
- Initial state bias eliminated by using `departure` $= 4.2$

  Even for a small number of jobs!

- The initial state bias that *seemed* to be evident before (16 replications) was natural small-sample variation
- Note: we were able to avoid initial state bias only because we already knew $E[W]$

Example 8.4.4: Increase the Number of Replications



- That departure $= 6.0$ seemed right was a false impression based on misleading small-sample statistics

Summary

- Want interval estimates for steady-state
- Replicated transient statistics can be used
- However, initial bias problem
- Need technique that avoids the initial bias problem

Method of Batch Means

- Previously, each replication was initialized with *same state*
- Gives initial bias problem
- *Batch means*:
  - Make *one* long run and partition into *batches*
  - Compute an average statistic for each batch
  - Construct an interval estimate using the batch means
- Initial state bias is eliminated
- State at the beginning of each batch is the state at the end of previous batch

Algorithm 8.4.1: Method of Batch Means

Algorithm 8.4.1

Consider a sequence of samples $x_1, x_2, \ldots, x_n$

1. Select a *batch size* $b > 1$
2. Group the sequence into k batches

$$\underbrace{x_1, x_2, \cdots, x_b}_{\text{batch 1}}, \underbrace{x_{b+1}, x_{b+2}, \ldots, x_{2b}}_{\text{batch 2}}, \underbrace{x_{2b+1}, x_{2b+2}, \ldots, x_{3b}}_{\text{batch 3}}, \ldots$$

   and for each calculate the *batch mean*

$$\bar{x}_j = \frac{1}{b} \sum_{i=1}^{b} x_{(j-1)b+i} \qquad j = 1, 2, \ldots, k$$

3. Compute $\bar{x}$ and s of batch means $\bar{x}_1, \bar{x}_2, \cdots, \bar{x}_k$

Algorithm 8.4.1: Method of Batch Means

Algorithm 8.4.1

4. Pick a *level of confidence* $1 - \alpha$ (typically $\alpha = 0.05$)
5. Calculate the critical value t^*=`idfStudent`$(k - 1, 1 - \alpha/2)$
6. Calculate the interval endpoints $\bar{x} \pm t^* s/\sqrt{k-1}$

- $(1 - \alpha) \times 100\%$ confident that the true *unknown* steady-state mean lies in the interval
- Provided b is large, true *even if the sample is autocorrelated*

Effect of Batch Parameters

- Provided no points are discarded:

$$\bar{x} = \frac{1}{k}\sum_{j=1}^{k}\bar{x}_j = \frac{1}{n}\sum_{i=1}^{n}x_i$$

- Choice of (b, k) has *no* impact on the *point* estimate
- Only the width of the *interval* estimate is affected

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Example 8.4.5: Effect of (b, k)

- Consider an initially idle $M/M/1$ with $(\lambda, \nu) = (1, 1.25)$
- Use `ssq2` to generate $n = 32768$ consecutive waits
- Using batch means with different (b, k):

(b, k)	$(8, 4096)$	$(64, 512)$	$(512, 64)$	$(4096, 8)$
$\bar{w}$	3.94 ± 0.11	3.94 ± 0.25	3.94 ± 0.29	3.94 ± 0.48

- Note that 3.94 is independent of (b, k)
- Width of the interval estimate is not

Is the Method of Batch Means Valid?

- For interval estimation, the batch means must be *iid Normal*
  1. Are the batch means *Normal*?

     As b increases, mean of b RVs tends to Normal
  2. Is the data actually independent?

     Autocorrelation (Section 4.4) becomes zero if b is large
- Therefore, as *b* increases, method of batch means becomes increasingly more valid

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Guidelines for Choosing (b, k)

- Note: If b is too large, k will be small giving wide interval estimates
- Number of batches k:
  - Avoid small-sample variation
  - $k \geq 32$; $k = 64$ is recommended
- Batch size b:
  - Want to ensure (approximate) independence
  - b should be at least twice the autocorrelation "cut-off" lag (Section 4.4)

Example 8.4.6: Achieving Independence

- Consider an $M/M/1$ with steady-state $\bar{x} = \lambda/\nu = 0.8$
- The effective cut-off lag is 128
- Choose $b = 256$ to ensure independent batch means
- Graph the autocorrelation function r_j versus the lag j

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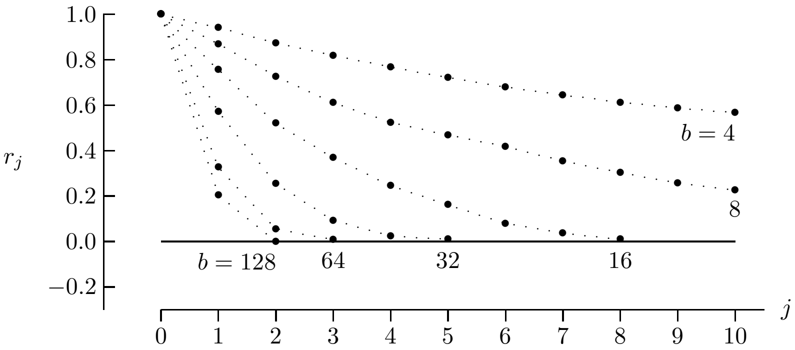
Example 8.4.6: Achieving Independence



- As b increases, autocorrelation is reduced
- For $b = 128$, no autocorrelation for $j > 1$

Example 8.4.6: Summary

- Consider $M/M/1$ with steady-state $\bar{x} = \lambda/\nu = 0.8$
- Use $k = 64$ batches of size $b = 256$
- The batch mean interval estimate will contain $E[W]$ with probability $1 - \alpha$
- $n = bk = 16384$ jobs required to produce this estimate
- If λ/ν were larger, n must be larger

Batch Means as a "Black Box"

- Can apply Algorithm 8.4.1 for *any* reasonable (b, k)
- Experiment with increasing (b, k) until "convergence"

  Set $k = 64$, and then successively double b

- Convergence: significant overlap in a sequence of increasingly more narrow interval estimates

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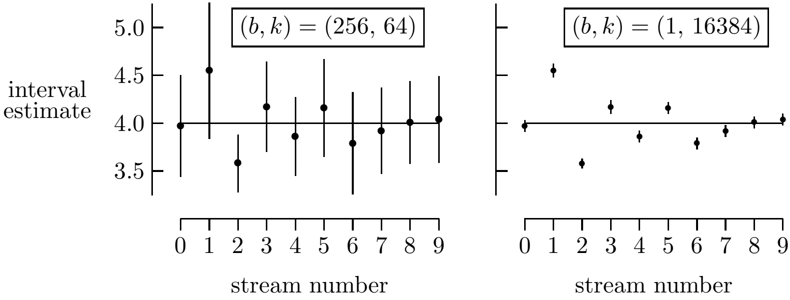
Example 8.4.7: "Black Box"

- Consider $M/M/1$ with $(\lambda, \nu) = (1.00, 1.25)$
- Produce 10 interval estimates using batch means



Example 8.4.7: "Black Box" Observations

- (256, 64): All but one brackets theoretical $E[W]$
- (1, 16384): Intervals are too small

    False sense of confidence (30% instead of 95%)
- Failure to correctly batch the data can lead to unrealistically small variance estimates
- Iteration: fix k and increase b until enough intervals overlap

    Remember that $n = bk$