Computer Science 616 - Stochastic Models in Computer Science
Fall 2007, Homework 6, due Nov 28 at noon

1. Let the transition probability matrix of a two-state Markov chain be given by

$$P = \begin{bmatrix} p & 1-p \\ 1-p & p \end{bmatrix},$$

with $0 < p < 1$. Prove that

$$P^{(n)} = \begin{bmatrix} \frac{1}{2} + \frac{1}{2}(2p-1)^n & \frac{1}{2} - \frac{1}{2}(2p-1)^n \\ \frac{1}{2} - \frac{1}{2}(2p-1)^n & \frac{1}{2} + \frac{1}{2}(2p-1)^n \end{bmatrix}$$

What is $\lim_{n\to\infty} P^{(n)}$? What does that mean? 20

2. Consider a DTMC with state space $S = \{0, \ldots, 4\}$, where $P_{0,4} = 1$ and, for $i > 0$, the DTMC has a uniform probability of going from state i to state $j = 0, \ldots, i-1$. Find the limiting probabilities. 50

3. A transition probability matrix P is a *stochastic* matrix, that is, the sum of the elements on each of its rows is one. If the sum of the entries in each column is also one, that is,

$$\forall j, \quad \sum_i P_{i,j} = 1,$$

the matrix is said to be doubly stochastic.

Show that, if a DTMC has a doubly stochastic transition probability matrix, if it is irreducible and aperiodic, and if it contains n states $\{1, \ldots, n\}$, its limiting probabilities are given by

$$\forall i, \quad \pi_i = \frac{1}{n}.$$

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4. (Section 4, #8) Suppose that a coin 1 has probability 0.7 of coming up heads, and coin 2 has probability 0.6 of coming up heads. If the coin flipped today comes up heads, then we select coin 1 to flip tomorrow, and, if it comes up tails, then we select coin 2 to flip tomorrow. If the coin initially flipped is equally likely to be coin 1 or 2, then what is the probability that the coin flipped on the third day after the initial flip is coin 1? 30

5. (Section 4, #4) Consider a process $\{X_n : n = 0, 1, 2, \ldots\}$ which takes on the values 0, 1, or 2. Suppose

$$P\{X_{n+1} = j | X_n = i, X_{n-1} = i_{n-1}, \ldots, X_0 = i_0\} = \begin{cases} P^I_{i,j} & \text{when } n \text{ is even} \\ P^{II}_{i,j} & \text{when } n \text{ is odd} \end{cases}$$

where $\sum_{j=0}^{2} P^I_{i,j} = \sum_{j=0}^{2} P^{II}_{i,j} = 1$, for $i \in \{0, 1, 2\}$. Is $\{X_n : n = 0, 1, 2, \ldots\}$ a Markov chain? If not, show how we may transform it into a Markov chain, by enlarging the state space 40

6. Three white and three black balls are distributed in two urns in such a way that each urn contains three balls. We say that the system is in state $i \in \{0, 1, 2, 3\}$ if the first urn contains i white balls. At each step, we draw one ball from each urn, and place the ball drawn from the first urn into the second urn, and the ball drawn from the second urn into the first urn. Let X_n denote the state of the system after the n-th step.

   - Explain why $\{X_n : n = 0, 1, 2, \ldots\}$ is a Markov chain, and calculate its transition probability matrix, giving full details of how you came up with your answers.
   - Compute the steady-state probability of being in each state (hint: the fastest way to answer this question is based on the idea of "flow balance" we described in class). 30

7. Consider a system operating in discrete (integral) time. Jobs arrive to the system and are serviced by the system, one at a time, as long as there are jobs to be serviced. Job interarrival times are *Geometric*(α) iid random variables. Job service times are *Geometric*(β) iid random variables. Interarrival and service times are independent of each other.

   (a) Model this system as a DTMC: specify its state space; specify its transition probability matrix, either graphically, or by defining the possible transitions out of each (type of) state.

   (b) Give the conditions under which the system is stable (i.e., the states are recurrent non-null), and justify your answer.

   (c) Compute the steady-state probability vector.

   (d) Verify that the DTMC is time-reversible. 40

8. (Section 4, #55, read Example 4.20 first!) Consider a population of individuals each of whom possesses two genes which can be either type A or type a. Supposes that in outward appearance type A is dominant and type a is recessive, that is, an individual will have the outward appearances of the recessive gene only if its pair is aa. Suppose that the population has stabilized, and the percentages of individuals having respective gene pairs AA, aa, and Aa are p, q, and r. Call an individual dominant or recessive depending on the outward characteristics it exhibits. Let S_{11} denote the probability that an offspring of two dominant parents will be recessive and let S_{10} denote the probability that an offspring of one dominant and one recessive parent will be recessive. Compute S_{11} and S_{10} to show that $S_{11} = S_{10}^2$. The quantities S_{10} and S_{11} are known in the literature as *Snyder's ratios.* 100