A Weak Bisimulation for Weighted Automata

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• Weighted Automata and Semirings
  • here focus on commutative & idempotent semirings
• Weak Bisimulation
• Composition operators
• Congruence property
Motivation

Notions of equivalence have been detected for many notations:
- process algebras
- automata
- stochastic processes

Equivalences are useful
- for a theoretical investigation of equivalent behaviour
- increasing the efficiency of analysis techniques by
  - minimization to the smallest equivalent automaton
  - composition of minimized automata
    requires congruence property!

Many different equivalences exist:
  trace-equivalence, failure equivalence, strong / weak bisimulation, ...

We consider a weak bisimulation for automata whose nodes and edges are
annotated by labels and weights.
Weights are elements of an algebra -> a semiring.
Semiring

- Semiring \( K_{+,*,0,1} = (K,+,*,0,1) \)
- Operations + and * defined for \( K \) have the following properties
  - associative: + and *
  - commutative: +
  - right/left distributive for + with respect to *
  - 0 and 1 are additive and multiplicative identities with \( 0 \neq 1 \)
  - for all \( k \in K \) \( 0 \ast k = k \ast 0 = 0 \)

- What is so special?
  - Similar to a ring, but each element need not(!) have an additive inverse.

- Special cases:
  - Idempotent semiring (or Dioid): + is idempotent: \( a+a=a \)
  - Commutative semiring: * is commutative
Semiring

Alternative definition

A semiring is a set $K$ equipped with two binary operations $+$ and $\cdot$, called addition and multiplication, such that:

- $(K, +)$ is a commutative monoid with identity element $0$:
  - $(a + b) + c = a + (b + c)$
  - $0 + a = a + 0 = a$
  - $a + b = b + a$

- $(K, \cdot)$ is a monoid with identity element $1$:
  - $(a \cdot b) \cdot c = a \cdot (b \cdot c)$
  - $1 \cdot a = a \cdot 1 = a$

- Multiplication distributes over addition:
  - $a \cdot (b + c) = (a \cdot b) + (a \cdot c)$
  - $(a + b) \cdot c = (a \cdot c) + (b \cdot c)$

- $0$ annihilates $K$:
  - $0 \cdot a = a \cdot 0 = 0$
Semiring

- Semiring $K_+, \ast = (K, +, \ast, 0, 1)$

- Examples
  - Boolean semiring $(B, \lor, \land, 0, 1)$
  - Real numbers $(R, +, \ast, 0, 1)$
  - max/+ semiring $(R \cup -\infty, \max, +, -\infty, 0)$
  - min/+ semiring $(R \cup \infty, \min, +, \infty, 0)$
  - max/min semiring $(R \cup -\infty \cup \infty, \max, \min, -\infty, \infty)$
  - square matrices $(R^{n \times n}, +, \ast, 0, 1)$

- A Kleene algebra is an idempotent semiring R with an additional unary operator $\ast : R \to R$ called the Kleene star. Kleene algebras are important in the theory of formal languages and regular expressions.
Idempotent Semiring

• Let’s define a partial order $\leq$ on an idempotent semiring: 
  \[ a \leq b \text{ whenever } a + b = b \]
  (or, equivalently, if there exists an $x$ such that $a + x = b$).

• Observations:
  - 0 is the least element with respect to this order:
    \[ 0 \leq a \text{ for all } a. \]
  - Addition and multiplication respect the ordering : 
    \[ a \leq b \text{ implies } \]
    \[ ac \leq bc \]
    \[ ca \leq cb \]
    \[ (a+c) \leq (b+c) \]
Kleene Algebra

A Kleene algebra is a set $A$ with two binary operations $+ : A \times A \rightarrow A$ and $\cdot : A \times A \rightarrow A$ and one function $* : A \rightarrow A$, (Notation: $a+b$, $ab$ and $a^*$) and

- Associativity of $+$ and $\cdot$, Commutativity of $+$
- Distributivity of $\cdot$ over $+$
- Identity elements for $+$ and $\cdot$:
  - exists $0$ in $A$ such that for all $a$ in $A$: $a + 0 = 0 + a = a$.
  - exists $1$ in $A$ such that for all $a$ in $A$: $a1 = 1a = a$.
- $a0 = 0a = 0$ for all $a$ in $A$.

The above axioms define a semiring.

We further require:
- $+$ is idempotent: $a + a = a$ for all $a$ in $A$. 
Kleene Algebra

- Let’s define a partial order $\leq$ on $A$:
  
  $a \leq b$ if and only if $a + b = b$
  
  (or equivalently: $a \leq b$ if and only if exists $x$ in $A$ such that $a + x = b$).

  With this order we can formulate the last two axioms about the operation $*$:

  - $1 + a(a^*) \leq a^*$ for all $a$ in $A$.
  - $1 + (a^*)a \leq a^*$ for all $a$ in $A$.
  - if $a$ and $x$ are in $A$ such that $ax \leq x$, then $a^*x \leq x$
  - if $a$ and $x$ are in $A$ such that $xa \leq x$, then $x(a^*) \leq x$

Think of

  $a + b$ as the "union" or the "least upper bound" of $a$ and $b$ and of
  
  $ab$ as some multiplication which is monotonic, in the sense that $a \leq b$
  
  implies $ax \leq bx$.

The idea behind the star operator is $a^* = 1 + a + aa + aaa + ...$

From the standpoint of programming theory, one may also interpret $+$ as "choice", $\cdot$ as "sequencing" and $*$ as "iteration".

- Example: Set of regular expressions over a finite alphabet
Weighted Automaton

A finite K-Automaton over finite alphabet $L$ (including $\tau$) is $A = (S, \alpha, T, \beta)$ with $S$ : finite set of states and maps giving initial, transition and final weights.

$$\alpha : S \rightarrow K,$$
$$T : S \times L \times S \rightarrow K,$$
$$\beta : S \rightarrow K$$

E.g. weights interpreted as costs, distances, time, ...
Weights multiply along a path, sum up over different paths.

We focus on commutative and idempotent K-automata, i.e., K is a semiring where * is commutative and + is idempotent!

Examples
- Boolean semiring
- $\max/+$ semiring
- $\min/+$ semiring
- $\max/min$ semiring

Transitions are described by matrices
Idempotency implies:
$$\sum_{k=0}^{\infty} A^k = \sum_{k=0}^{\infty} A^k \cdot \sum_{k=0}^{\infty} A^k$$
Examples

- **Boolean semiring**, 
  - weights encode existence / non-existence of paths in directed graphs
  - labels serve the same purpose, hence weights are usually omitted
  - idempotency is quite natural:
    - existence of a paths remains valid in case of multiple paths
- **Max/+ semiring**
  - interpretation
    - weights are multiplied along a path, * is +, weight of a path is the sum over all edge weights
    - sum over all paths starting at a node is given by max, hence the path with highest weight is taken (snob if these are costs, greedy if this is profit)
- **Max/Min semiring**
  - interpretation
    - weight of a path: * is min, weight of a path gives minimal weight of its edges
    - sum over paths: + is max, selects path whose bottleneck has largest capacity
Some more notation

- Weight of path $\pi$
  
  $$w(\pi) = \alpha(s_0) \cdot \left( \prod_{i=1}^{n} T(s_{i-1}, l_i, s_i) \right) \beta(s_n)$$
  
  or by vectors/matrices
  
  $$= a(s_0) \left( \prod_{i=1}^{n} M_{li}(s_{i-1}, s_i) \right) b(s_n)$$

- Weight of sequence $\sigma$
  
  $$w(\sigma) = a \cdot \left( \prod_{i=1}^{n} M_{li} \right) b$$

- Define automaton $A^*$ where sequences of $\tau$-transitions are replaced by single $\varepsilon$ transition.
  
  $$M_{\varepsilon} = M^* = \sum_{i=0}^{\infty} M_{\tau}^i,$$
  
  $$M_l' = M_{\varepsilon} \cdot M_l \cdot M_{\varepsilon}^{'}, \quad b' = M_{\varepsilon} \cdot b$$

- Weight of sequence $\sigma'$
  
  $$w'(\sigma') = a \cdot \left( \prod_{i=1}^{n} M_{li}' \right) b'$$
  
  $$= a \cdot \left( \prod_{i=1}^{n} M_{\varepsilon} \cdot M_{li} \cdot M_{\varepsilon}' \right) M_{\varepsilon} \cdot b$$
  
  $$= a \cdot \left( \prod_{i=1}^{n} M_{\varepsilon} \cdot M_{li} \right) M_{\varepsilon} \cdot b$$
Weak bisimulation of K-automata

An equivalence relation $R \subseteq S \times S$ is a weak bisimulation relation if for all $(s_1, s_2) \in R$, all $l \in L \setminus \{\tau\} \cup \{\varepsilon\}$, all equivalence classes $C \in S / R$

\[
\begin{align*}
\alpha(s_1) &= \alpha(s_2) \quad \text{or} \quad a(s_1) = a(s_2) \\
\beta'(s_1) &= \beta'(s_2) \quad \text{in terms of matrices} \\
T'(s_1, l, C) &= T'(s_2, l, C) \\
M'_l(s_1, C) &= M'_l(s_2, C)
\end{align*}
\]

Two states are weakly bisimilar, $s_1 \approx s_2$, if $(s_1, s_2) \in R$

Two automata are weakly bisimilar, $A_1 \approx A_2$, if there is a weak bisimulation on the union of both automata such that $\alpha(C_1) = \alpha(C_2)$ for all $C \in S / R$
Theorem

If \( A_1 \approx A_2 \) for Ki - Automata \( A_1, A_2 \) then \( w_1 \sigma = w_2 \sigma \) for all \( \sigma \in L' \) where \( L' = (L_1 \cup L_2) \setminus \{\tau\} \cup \{\varepsilon\} \)

Weights of sequences are equal in weakly bisimilar automata.

Ki is commutative and idempotent semiring K

Sequence? sequence considers all paths that have same sequence of labels, may start or stop at any state

Weakly? Paths can contain subpaths of \( \tau \)-labeled transitions represented by a single \( \varepsilon \)-labeled transition.
Example

is weakly bisimilar for \( \max/+ \) semiring

is weakly bisimilar for \( \min/\max \) semiring

is weakly bisimilar for \( \min/+ \) and \( \max/\min \) semiring
Deloping further

- consider largest bisimulation, i.e. the one with fewest classes
  - same argumentation as for Milner’s CCS
- computation by O(nm) fix point algorithm, n states, m edges
  - starting from boolean semiring as in the concurrency workbench (Cleaveland, Parrows, Steffen)
  - extended to semiring of real numbers by Buchholz
  - extension to more general semirings straightforward
  - more efficient ones like O(n log m) as for boolean semiring ???
  - presupposes also computation of A*

- bisimulation useful if preserved by composition operations (congruence property)
  - composition operations for automata ?
    sum
direct or cascaded product
synchronized product
specific type of choice

  good news: these are all ok !!!
  but how are they defined ?
Composition operations

- **Sum**
  - union of automata with no interaction

- **Direct or cascaded product**
  - build union of state sets and labels
  - take initial weights only from first automaton
  - take final weights only from second automaton
  - connect first with second automaton by new $\tau$-transitions between final states of first, initial states of second automaton

![Diagram showing composition operations](image-url)
Composition operations

- **Sum**
  - union of automata with no interaction

- **Direct or cascaded product**
  - build union of state sets and labels
  - take initial weights only from first automaton
  - take final weights only from second automaton
  - connect first with second automaton by new tau-transitions between final states of first, initial states of second automaton

![Diagram showing composition operations]
Composition operations

- Synchronized product (with subset of labels for synchronisation)
  - build cross product of state sets, union of label sets
  - take product of initial weights
  - take product of final weights
  - take product of transition weights in case of synch otherwise proceed independently
  - Note: free product is special case with empty set of labels for synchronisation
Composition operations

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• Choice
  – connects automata by merging only initial states
  – initial states must be unique and have initial weight 1 and equal final weights
Theorem

If $A_1 \approx A_2$ and $A_3$ are finite Ki - Automata then
1. $A_1 + A_3 \approx A_2 + A_3$
2. $A_1 \cdot A_3 \approx A_2 \cdot A_3$ and $A_2 \cdot A_3 \approx A_1 \cdot A_3$
3. $A_1 \parallel_{LC} A_3 \approx A_2 \parallel_{LC} A_3$ and $A_3 \parallel_{LC} A_1 \approx A_3 \parallel_{LC} A_2$
and if choice is defined then
4. $A_1 \lor A_3 \approx A_2 \lor A_3$ and $A_3 \lor A_1 \approx A_3 \lor A_2$

Some notes on proofs:
• proofs are lengthy,
• argumentation based matrices helps,
• argumentation along paths, resp. sequences more tedious
• idempotency simplifies valuation for concatenation of $\tau^*|\tau^*$ transitions
• note that algebra does not provide inverse elements wrt + and *
Summary

• Weak Bisimulation for weighted automata over commutative and idempotent semirings
• Congruence for
  – sum
  – direct or cascaded product
  – synchronized product
  – specific choice operator

References: