The Average Distance between Nodes in the Cyclic Tower of Hanoi Digraph

Paul K. Stockmeyer
College of William and Mary

Abstract

The cyclic Tower of Hanoi puzzle is similar to the traditional tower puzzle, but with the added restriction that all disks move between pegs in a clockwise direction, from peg A to B, from B to C, or from C to A. Many aspects of this puzzle have been analyzed, including the average distance in the state digraph from a random state to a designated goal state in which all disks are on one peg. Using similar but somewhat cleaner methods, we extend this result by computing the mean and variance of the distance between a random pair of states. Our results are compared to analogous ones for the traditional Tower of Hanoi puzzle.

1 Introduction and Background

The Tower of Hanoi

The famous Tower of Hanoi puzzle, invented in 1883 by the French mathematician Édouard Lucas [8], consists of three pegs, usually designated A, B, and C, and a set of n pierced disks of differing diameters that can be stacked on the pegs. By convention the disks are numbered from 1 to n in increasing order of size, and a tower is formed by stacking the disks on one of the pegs, in order, with disk n on the bottom and disk 1 on the top. The challenge is to transport the tower to another peg, moving the disks from peg to peg one at a time, without ever placing a disk on top of a smaller one. It is well known that $2^n - 1$ moves are necessary and sufficient for performing this task.

All early solutions to this puzzle were described recursively: to transport a tower of n disks from A to C, say, we transport the subtower consisting of the top $n - 1$ disks from A to B; then move disk n from A to C; and then transport the subtower from B to C. Although compact and elegant, this recipe is not particularly helpful to humans actually trying to carry out the moves necessary to solve the puzzle. A variety of iterative algorithms have been developed that generate the minimal move sequence in a more user friendly manner.

The Cyclic Puzzle

The cyclic tower of Hanoi puzzle was invented in 1981 by Atkinson [1] in an attempt to find a variation for which no simple iterative solution could be devised. The fact that several iterative algorithms were soon discovered for this version—see [3] and [11]—does not diminish its attractiveness.

In the cyclic tower puzzle, the pegs are assumed to be arranged in a triangle, with clockwise ordering A, B, C. The disks are restricted to moving from peg to peg in a clockwise direction, from A to B, from B to C, or from C to A. Atkinson considers two problems: to transport a tower of n disks one step clockwise, which we call the *short* problem; and to transport a tower of n disks one step counterclockwise, which we call the *long* problem. His elegant solution consists of two interdependent recursive procedures. Stripped to their bare essentials they are given in Figure 1.

```
procedure short(n);
  if (n > 0) then
     long(n-1);
     move disk n;
     long(n-1);
```

```
procedure long(n);
  if (n > 0) then
     long(n-1);
     move disk n;
     short(n-1);
     move disk n;
     long(n-1);
```

Figure 1

Of course an actual computer program might want to specify the source peg as a parameter, and print out a list of moves giving disk number, source peg, and perhaps destination peg. We are primarily interested here in the number of disk moves these procedures generate. If we let

$S(n)$ and $L(n)$ denote the number of disk moves made by the procedures `short(n)` and `long(n)`, respectively, we see immediately that

$$\begin{aligned} S(n) &= 2L(n-1)+1 \quad \text{and} \\ L(n) &= 2L(n-1)+S(n-1)+2 \end{aligned}$$

for $n \geq 1$, with $S(0) = L(0) = 0$. Standard techniques yield the solutions

$$\begin{aligned} S(n) &= \frac{3+\sqrt{3}}{6}\left(1+\sqrt{3}\right)^n + \frac{3-\sqrt{3}}{6}\left(1-\sqrt{3}\right)^n - 1 \quad \text{and} \\ L(n) &= \frac{3+2\sqrt{3}}{6}\left(1+\sqrt{3}\right)^n + \frac{3-2\sqrt{3}}{6}\left(1-\sqrt{3}\right)^n - 1, \end{aligned}$$

which can be confirmed by induction.

Atkinson did not provide a proof that his two procedures produce minimal length move sequences, perhaps believing that it was obvious. We will prove a stronger and somewhat surprising result as Fact 3 in a later section.

Generalizations

Several authors have suggested generalizing the Tower of Hanoi puzzle by allowing the starting configuration to be any distribution of disks to pegs, with the disks on each peg stacked in order. As before, the challenge is to form the disks into a tower on a designated peg, following the standard rules for moves. This generalization was extended to the cyclic puzzle by Er [4], who developed minimal move algorithms for solving the cyclic puzzle from an arbitrary starting distribution of disks. Subsequently, Er [5] computed the average number $A(n)$ of moves needed to transform a random initial *proper* configuration (with the disks stacked in order on their assigned pegs) to a designated *goal* configuration (with all n disks stacked on a designated peg). We present an alternative derivation of his result later as Theorem 1.

The main purpose of this paper is to extend this result to the case where both the initial configuration and the final configuration are randomly chosen. Informally speaking, we answer the question "how many (cyclic) moves apart, on the average, are two randomly chosen configurations?" This is analogous to the result of Hinz [6] and Chan [2] for the standard tower puzzle. Along the way we make some interesting observations and relate the cyclic case to the standard case when appropriate.

2 The State Digraph

Definitions and Conventions

The idea of a *state graph* for the Tower of Hanoi puzzle originated with Scorer, Grundy, and Smith [9]. In this graph, the vertices, or nodes, are the 3^n proper states or configurations of the tower puzzle, and the edges are the $(3^{n+1} - 3)/2$ allowable moves between states. It is well known that for any number n of disks, the state graph is a planar graph that can be embedded in an equilateral triangle with all edges of equal length. With this embedding, the graph approaches the familiar fractal known as the Sierpiński Gasket (see [7], [10]) as n increases. For the cyclic puzzle, the state graph becomes a digraph, with each arc oriented in the direction representing a clockwise disk move. Although not absolutely essential for understanding what follows, the state digraph makes the concepts and theorems of graph theory available to us, and provides us with a useful visualization.

We observe certain conventions in drawing the state digraph for the cyclic tower puzzle. Each vertex is labeled with a string of length n from the set $\{A, B, C\}$, naming the pegs that the disks occupy in that state, *from the largest to the smallest.* We place vertex $AA\ldots A$ in the lower left corner, vertex $BB\ldots B$ at the top, and vertex $CC\ldots C$ in the lower right corner. We call these three vertices the *corner* vertices, representing what we called *goal* states above, or what some authors call *perfect* states. The digraph consists of three major subdigraphs, each isomorphic to the digraph of the puzzle with one fewer disk, joined by arcs we call *links*. The A subdigraph in the lower left, for example, contains all states for which the largest disk is on peg A. It is joined to the B subdigraph by the $A \to B$ link from vertex $AC\ldots C$ to vertex $BC\ldots C$, representing a move of disk n from peg A to peg B. The B subdigraph is joined to the C subdigraph by the $B \to C$ link from vertex $BA\ldots A$ to vertex $CA\ldots A$, representing a move of disk n from peg B to peg C. Figure 2, which displays the state digraph for the cyclic puzzle with 4 disks, should make our conventions and nomenclature clear.

Figure 2

Following standard graph theory terminology, a *path* is a sequence $v_0, v_1, \ldots, v_k$ of distinct vertices where vertex v_{i-1} is adjacent to vertex v_i for all i from 1 to k. The *length* of the path is k, and the *distance* from one vertex to another is the length of the shortest path from the first to the second. The reader is invited to trace the path of length $L(4) = 59$ from $AAAA$ to $CCCC$ and the path of length $S(4) = 43$ from $CCCC$ to $AAAA$ in Figure 2.

Some Simple Facts

In this section we note several useful facts about the state digraph, all easily proved. Some are so intuitively clear that people tend to overlook the fact that a proof is warranted, while others are surprisingly counterintuitive.

Fact 1. The state digraph for the cyclic puzzle is strongly connected: there is a path joining any given vertex to any other.

Proof. This is easily proved by induction on n. If n is 0 there is nothing to prove, while if the two vertices are in the same major subdigraph, the result follows immediately from the induction hypothesis. Otherwise, the induction hypothesis allows us to piece together the desired path from subpaths in two or three of the major subdigraphs, connected by one or two links.

Fact 2. Rotating the state digraph through ± 120 degrees constitutes a digraph automorphism. Reflecting it in a line through a corner vertex and the opposite link creates an anti-automorphism, mapping the digraph onto its converse.

Proof. A rotation corresponds to a cyclic relabeling of the pegs, which has no effect on the allowable moves. A reflection corresponds to interchanging the labels on two of the pegs, which reverses the allowable direction of every move.

One consequence of this fact, which we shall use later, is that the average distance $A(n)$ from a random vertex to a specified corner vertex is the same as the average distance from a specified corner vertex to a random vertex. This is so despite the fact that the distance from a vertex v to a corner vertex is generally not equal to the distance from the corner vertex to v.

Fact 3. There is only one path from any given corner vertex to another corner vertex.

Proof. From Fact 2, we may assume that the first corner is vertex $AA\ldots A$. A path from this corner to the corner vertex $BB\ldots B$ must use the $A \to B$ link exactly once, and no others. By induction, we know that there is only one path in the A subdigraph from vertex $AA\ldots A$ to the beginning of the $A \to B$ link at $AC\ldots C$, and only one path in the B subdigraph from the end of the $A \to B$ link at $BC\ldots C$ to the corner vertex $BB\ldots B$. These two paths, together with the $A \to B$ link, form the unique path from $AA\ldots A$ to $BB\ldots B$. Similarly, the one path from $AA\ldots A$ to $CC\ldots C$ consists of the unique path from $AA\ldots A$ to $AC\ldots C$ in the A subdigraph, the $A \to B$ link, the unique path from $BC\ldots C$ to $BA\ldots A$ in the B subdigraph, the $B \to C$ link, and the unique path from $CA\ldots A$ to $CC\ldots C$ in the C subdigraph.

In terms of disk moves, we see that the move sequences generated by the procedures `short(n)` and `long(n)` are not just *minimal* solution sequences, they are in fact the *only* move sequences that solve their corresponding problems without ever repeating a puzzle state. This is in sharp contrast to the case with the standard puzzle, for which there exist non-repeating solution sequences, i.e. corner to corner paths, of all lengths from $2^n - 1$ to $3^n - 1$.

Fact 4. For any two states in the same major subdigraph, a minimal length path from one to the other lies entirely within that major subdigraph.

Proof. Let v_1 and v_2 be any two vertices in the A subdigraph. A path from v_1 to v_2 is either entirely within the A subdigraph or else it crosses all three links. In the second case, the path contains a subpath from $AC\dots C$ to $AB\dots B$ that crosses the three links and passes through the B and C subdigraphs. This subpath has length $1+S(n-1)+1+S(n-1)+1 = 2S(n-1)+3$. But there is a shorter path from $AC\dots C$ to $AB\dots B$, entirely within subdigraph A, of length $L(n-1)$. Thus a path crossing all three links can not be minimal.

Fact 5. The shortest path from any one vertex to any other is unique.

The proof is similar to those given above, and is left to the reader. We note that the analogous statement for the standard puzzle is false. In the standard puzzle there are pairs of vertices, for example, that are connected by one minimal path containing one link, and a second minimal path containing the other two links.

Fact 6. The maximum distance between any two vertices is $L(n)$. This distance only occurs between corner vertices.

This proof is also left to the reader. Fact 6 asserts that the long problem is the hardest problem that can be posed for the cyclic puzzle, at least in terms of the required number of moves between proper states. In the case of the standard puzzle, the corner to corner distance of $2^n - 1$ is also maximal, but not uniquely so—there are many other pairs of vertices that are this far apart.

3 Main Results

We are now in a position to state and prove the main results of this paper. Although the theorems are stated in terms of paths and distance in the state digraph, the reader should have no trouble translating them into statements about sequences of disk moves. We begin with an alternative derivation of the result of Er [5] mentioned earlier.

Theorem 1. *(Er) Let v be a vertex selected from a uniform distribution over the vertices of the n-disk cyclic Tower of Hanoi digraph, and let v_0 be a designated corner vertex. Then the average value $A(n)$ of* *the distance* $D(v, v_0)$ *from* v *to* v_0 *satisfies the recurrence relation*

$$A(n) = A(n-1) + \frac{1}{3}\Big(L(n) + 1\Big)$$

for $n \geq 1$, *with* $A(0) = 0$. *The solution to this recurrence is*

$$A(n) = \frac{5+3\sqrt{3}}{18}\left(1+\sqrt{3}\right)^n + \frac{5-3\sqrt{3}}{18}\left(1-\sqrt{3}\right)^n - \frac{5}{9}.$$

Proof. We can assume without loss of generality that v_0 is the corner vertex $CC\ldots C$. The derivation considers three cases, depending on the location of v. We have

$$\begin{aligned}
A(n) &= \frac{1}{3^n}\Big(\sum_{v\in A} D(v,v_0) + \sum_{v\in B} D(v,v_0) + \sum_{v\in C} D(v,v_0)\Big) \\
&= \frac{1}{3^n}\Big(\Big(3^{n-1}\Big)\Big(A(n-1) + 1 + S(n-1) + 1 + L(n-1)\Big) \\
&\qquad + \Big(3^{n-1}\Big)\Big(A(n-1) + 1 + L(n-1)\Big) \\
&\qquad + \Big(3^{n-1}\Big)A(n-1)\Big) \\
&= \frac{1}{3}\Big(3A(n-1) + 2L(n-1) + S(n-1) + 3\Big) \\
&= A(n-1) + \frac{1}{3}\Big(L(n) + 1\Big).
\end{aligned}$$

This recurrence is easily solved by standard methods, using the expression for $L(n)$ given earlier. Alternatively, the formula for the solution can be confirmed by induction.

We now generalize this result to the case of two random vertices.

Theorem 2. *Let* v_1 *and* v_2 *be independently chosen, uniformly distributed vertices in the* n*-disk cyclic Tower of Hanoi digraph. Then the mean value* $M(n)$ *of the distance* $D(v_1, v_2)$ *satisfies the recurrence relation*

$$M(n) = \frac{1}{3}\Big(M(n-1) + 4A(n-1) + S(n-1)\Big) + 1$$

for $n \geq 1$, *with* $M(0) = 0$. *The solution to this relation is*

$$\begin{aligned}
M(n) &= \frac{77+57\sqrt{3}}{414}\left(1+\sqrt{3}\right)^n + \frac{77-57\sqrt{3}}{414}\left(1-\sqrt{3}\right)^n \\
&\qquad - \frac{6}{23}(1/3)^n - \frac{1}{9}.
\end{aligned}$$

Proof. We can assume without loss of generality that v_2 is in the C subdigraph. Following the same scheme as in the proof of Theorem 1, we have

$$\begin{aligned} M(n) &= \frac{3}{(3^n)^2}\Big(\sum_{\substack{v_1\in A\\ v_2\in C}} D(v_1,v_2) + \sum_{\substack{v_1\in B\\ v_2\in C}} D(v_1,v_2) + \sum_{\substack{v_1\in C\\ v_2\in C}} D(v_1,v_2)\Big) \\ &= \frac{3}{(3^n)^2}\Big(\left(3^{n-1}\right)^2 \Big(A(n-1)+1+S(n-1)+1+A(n-1)\Big) \\ &\qquad + \left(3^{n-1}\right)^2 \Big(A(n-1)+1+A(n-1)\Big) \\ &\qquad + \left(3^{n-1}\right)^2 M(n-1)\Big) \\ &= \frac{1}{3}\Big(M(n-1)+4A(n-1)+S(n-1)+3\Big). \end{aligned}$$

As before, this recurrence can be solved by standard methods, or the formula for the solution can be verified by induction.

With significantly more effort, we can also obtain the variances for the distances $D(v,v_0)$ and $D(v_1,v_2)$. Space requirements prevent the inclusion of the proofs, but sketches of the derivations will be available at the web site `http://www.cs.wm.edu/~pkstoc/toh.html`.

Theorem 3. *The variance V_1 of the distance $D(v,v_0)$ from a random vertex to a specified corner vertex of the n-disk cyclic Tower of Hanoi digraph has the value*

$$\begin{aligned} V_1(n) &= \frac{6+5\sqrt{3}}{162}\left(1+\sqrt{3}\right)^{2n} + \frac{6-5\sqrt{3}}{162}\left(1-\sqrt{3}\right)^{2n} \\ &\quad -\frac{2}{81}(-2)^n - \frac{4}{81}. \end{aligned}$$

The analogous variance V_2 of the distance $D(v_1,v_2)$ between two random vertices has the value

$$\begin{aligned} V_2(n) &= \frac{5127+19346\sqrt{3}}{557037}\left(1+\sqrt{3}\right)^{2n} + \frac{5127-19346\sqrt{3}}{557037}\left(1-\sqrt{3}\right)^{2n} \\ &\quad + \frac{154+114\sqrt{3}}{1587}\left(\frac{1+\sqrt{3}}{3}\right)^{n} + \frac{154-114\sqrt{3}}{1587}\left(\frac{1-\sqrt{3}}{3}\right)^{n} \\ &\quad - \frac{36}{529}\left(\frac{1}{3}\right)^{2n} - \frac{2}{273}\left(\frac{1}{3}\right)^{n} - \frac{500}{13041}(-2)^n - \frac{8}{81}. \end{aligned}$$

References

[1] Atkinson, M. D., The cyclic towers of Hanoi, *Information Processing Letters* **13** (1981), 118–119.

[2] Chan, Tat-Hung, A statistical analysis of the towers of Hanoi problem, *Intern. J. Computer Math.* **28** (1989), 57–65.

[3] Er. M. C., An iterative algorithm for the cyclic towers of Hanoi problem, *Intern. J. Comput. Inform. Sci.* **13** (1984), 123–129.

[4] Er. M. C., A generalization of the cyclic towers of Hanoi: an iterative solution, *Intern. J. Computer Math.* **15** (1984) 129–140.

[5] Er, M. C., The complexity of the generalised cyclic towers of Hanoi problem, *Journal of Algorithms* **6** (1985), 351–358.

[6] Hinz, Andreas M., The tower of Hanoi, *L'Enseignement Mathématique* **35** (1989), 289–321.

[7] Hinz, Andreas M. & Schief, Andreas, The average distance on the Sierpiński gasket, *Probability Theory and Related Fields* **87** (1990), 129–138.

[8] Lucas, Édouard, *Récréations Mathématiques, vol. III,* Gauthier-Villars, Paris, 1893. Reprinted several times by Albert Blanchard, Paris.

[9] Scorer, R. S, Grundy, P. M., & Smith, C. A. B., Some binary games, *Math. Gazette* **280** (1944), 96–103.

[10] Sierpiński, W., Sur une courbe dont tout point est un point de ramification, *C. R. Acad. Sci. Paris* **160** (1915), 302–305.

[11] Walsh, T. R., Iteration strikes back—at the cyclic Towers of Hanoi, *Information Processing Letters* **16** (1983), 91–93.