The Foundations: Logic and Proofs

Chapter 1, Part II: Predicate Logic

With Question/Answer Animations

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Summary

Predicate Logic (First-Order Logic (FOL), Predicate Calculus)

- The Language of Quantifiers
- Logical Equivalences
- Nested Quantifiers
- Translation from Predicate Logic to English
- Translation from English to Predicate Logic

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Predicates and Quantifiers

Section 1.4

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Section Summary

Predicates

Variables

Quantifiers

- Universal Quantifier
- Existential Quantifier

Negating Quantifiers

- De Morgan's Laws for Quantifiers

Translating English to Logic

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Propositional Logic Not Enough

- if we have:
  - "All men are mortal."
  - "Socrates is a man."
  - does it follow that "Socrates is mortal?"
  - this can't be represented in propositional logic
    - need a language that talks about objects, their properties, and their relations

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Introducing Predicate Logic

- predicate logic uses the following new features:
  - variables: x, y, z
  - predicates: P(x), M(x)
  - quantifiers
- propositional functions are a generalization of propositions
  - they contain variables and a predicate, e.g., P(x)
  - variables can be replaced by elements from their domain

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Propositional Functions

- propositional functions become propositions (and have truth values) when their variables are each replaced by a value from the domain (or bound by a quantifier, as we will see later).
- the statement P(x) is said to be the value of the propositional function P at x
- example: let P(x) denote x > 0 with a domain of integers
  - P(-3) is false
  - P(0) is false
  - P(3) is true
- often the domain is denoted by U, so in this example U is the integers

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Examples of Propositional Functions

- let R(x, y, z): x + y = z with U (for all three variables) as the integers
  - R(2,-1,5)
    - **Solution: F**
  - R(3,4,7)
    - **Solution: T**
  - R(x, 3, z)
    - **Solution: Not a Proposition**
- let Q(x, y, z): x - y = z, with U as the integers
  - Q(2,-1,3)
    - **Solution: T**
  - Q(3,4,7)
    - **Solution: F**
  - Q(x, 3, z)
    - **Solution: Not a Proposition**

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Compound Expressions

- connectives from propositional logic carry over to predicate logic
- if *P(x)* denotes "*x* > 0," find these truth values:
  - P(3) ∨ P(-1) **Solution**: T
  - P(3) ∧ P(-1) **Solution**: F
  - P(3) → P(-1) **Solution**: F
  - P(3) → ¬P(-1) **Solution**: T
- expressions with variables are not propositions and therefore do not have truth values; for example,

  P(3) ∧ P(*y*)

  P(*x*) → P(*y*)
- when used with quantifiers, these expressions (propositional functions) become propositions

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Quantifiers

Charles Pierce (1839-1914)

- quantifiers express the meaning of English words including all and some
  - All men are Mortal.
  - Some cats do not have fur.
- the two most important quantifiers
  - universal quantifier: "For all", symbol: ∀
  - existential quantifier, "There exists", symbol: ∃
- in ∀x P(x) and ∃x P(x)
  - ∀x P(x) asserts P(x) is true for <u>every</u> x in the domain.
  - ∃x P(x) asserts P(x) is true for <u>some</u> x in the domain.
- quantifiers bind the variable x in these expressions

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Universal Quantifier

∀x P(x) is read as "For all x, P(x)" or "For every x, P(x)"

Examples:

1) if P(x) denotes "x > 0" and U is the integers,
   ∀x P(x) is F
2) if P(x) denotes "x > 0" and U is the positive integers
   ∀x P(x) is T
3) if P(x) denotes "x is even" and U is the integers
   ∀x P(x) is false

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Existential Quantifier

- ∃x P(x) is read as "For some x, P(x)," or "There is an x such that P(x)," or "For at least one x, P(x)."

Examples:

1. if P(x) denotes "x > 0" and U is the integers,
   ∃x P(x) is T
   it is also T if U is the positive integers
2. if P(x) denotes "x < 0" and U is the positive integers,
   ∃x P(x) is F
3. if P(x) denotes "x is even" and U is the integers,
   ∃x P(x) is T

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Thinking about Quantifiers

we can think of quantification as looping through the elements of the domain

to evaluate ∀x P(x), loop through all x in the domain

- if at every step P(x) is T, then ∀x P(x) is T
- if at some step P(x) is F, then ∀x P(x) is F and the loop terminates

to evaluate ∃x P(x), loop through all x in the domain

- if at some step, P(x) is T, then ∃x P(x) is T and the loop terminates
- if the loop ends without finding an x for which P(x) is T, then ∃x P(x) is F

works if domains are infinite, but loops may not terminate

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Properties of Quantifiers

the truth value of ∃x P(x) and ∀x P(x) depend on both the propositional function P(x) and on the domain U

Examples:

1. if U is the positive integers and P(x) is the statement $x < 2$, then ∃x P(x) is T, but ∀x P(x) is F
2. if U is the negative integers and P(x) is the statement $x < 2$, then both ∃x P(x) and ∀x P(x) are T
3. If U consists of 3, 4, and 5, and P(x) is the statement $x > 2$, then both ∃x P(x) and ∀x P(x) are T, but if P(x) is the statement $x < 2$, then both ∃x P(x) and ∀x P(x) are F

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Precedence of Quantifiers

- the quantifiers ∀ and ∃ have higher precedence than all the logical operators
- example: ∀x P(x) ∨ Q(x) means (∀x P(x)) ∨ Q(x)

  ∀x (P(x) ∨ Q(x)) means something different
- unfortunately, people write ∀x P(x) ∨ Q(x) when they mean ∀x (P(x) ∨ Q(x))

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Translating from English to Logic

Example 1: translate the following sentence into predicate logic:
Every student in this class has taken a course in Java.

Solution:

first decide on the domain U

Solution 1: if U is all students in this class, define a propositional function J(x) denoting "x has taken a course in Java" and translate as ∀x J(x)

Solution 2: if U is all people, define a propositional function S(x) denoting "x is a student in this class" and translate as ∀x (S(x)→ J(x))

∀x (S(x) ∧ J(x)) is not correct (what does it mean?)

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Translating from English to Logic

Example 2: translate the following sentence into predicate logic:
Some student in this class has taken a course in Java.

Solution:

first decide on the domain U

Solution 1: if U is all students in this class, translate as

∃x J(x)

Solution 2: if U is all people, then translate as
∃x (S(x) ∧ J(x))

∃x (S(x) → J(x)) is not correct (what does it mean?)

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Back to the Socrates Example

- introduce the propositional functions

  Man (x): x is a man
  Mortal (x): x is mortal
- specify the domain as all people
- the two premises are represented as

  ∀x (Man(x) → Mortal(x))
  Man (Socrates)
- the conclusion is

  Mortal (Socrates)
- later we will show how to prove that the conclusion follows from the premises

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Equivalences in Predicate Logic

statements involving predicates and quantifiers are logically equivalent if and only if they have the same truth value

- for every predicate substituted into these statements and
- for every domain of discourse used for the variables in the expressions.

the notation $S \equiv T$ indicates that S and T are logically equivalent.

Example: $\forall x \neg\neg S(x) \equiv \forall x\, S(x)$

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Thinking about Quantifiers as Conjunctions and Disjunctions

- a universally quantified proposition is equivalent to a conjunction of propositions without quantifiers
- an existentially quantified proposition is equivalent to a disjunction of propositions without quantifiers
- if U consists of the integers 1,2, and 3:

$$\forall x\, P(x) \equiv P(1) \wedge P(2) \wedge P(3)$$

$$\exists x\, P(x) \equiv P(1) \vee P(2) \vee P(3)$$

- even if the domains are infinite, you can still think of the quantifiers in this fashion, but the equivalent expressions without quantifiers will be infinitely long

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Negating Quantified Expressions

- consider $\forall x\, J(x)$:

  Every student in your class has taken a course in Java.

  here J(x): "x has taken a course in Java" and the domain is students in your class

- negating the original statement gives
  It is not the case that every student in your class has taken Java.

  this implies that
  There is a student in your class who has not taken Java.

- symbolically $\neg\forall x\, J(x)$ and $\exists x\, \neg J(x)$ are equivalent

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Negating Quantified Expressions

- now Consider $\exists x\, J(x)$:

  There is a student in this class who has taken a course in Java."

  where J(x): x has taken a course in Java

- negating the original statement gives
  It is not the case that there is a student in this class who has taken Java.

  this implies that
  Every student in this class has not taken Java

- symbolically $\neg\exists x\, J(x)$ and $\forall x\, \neg J(x)$ are equivalent

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De Morgan's Laws for Quantifiers

The rules for negating quantifiers are as follows:

TABLE 2 De Morgan's Laws for Quantifiers.

Negation	*Equivalent Statement*	*When Is Negation True?*	*When False?*
$\neg\exists x P(x)$	$\forall x \neg P(x)$	For every *x*, *P*(*x*) is false.	There is an *x* for which *P*(*x*) is true.
$\neg\forall x P(x)$	$\exists x \neg P(x)$	There is an *x* for which *P*(*x*) is false.	*P*(*x*) is true for every *x*.

the reasoning in the table shows that

$$\neg\forall x P(x) \equiv \exists x \neg P(x)$$

$$\neg\exists x P(x) \equiv \forall x \neg P(x)$$

these are important – you will use these

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Translation from English to Logic

Examples:

1. "Some student in this class has visited Mexico."

   Solution: Let M(x) denote "x has visited Mexico" and S(x) denote "x is a student in this class," and U be all people.

   $$\exists x\, (S(x) \wedge M(x))$$

2. "Every student in this class has visited Canada or Mexico."

   Solution: Add *C*(x) denoting "*x* has visited Canada."

   $$\forall x\, (S(x) \rightarrow (M(x) \vee C(x)))$$

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Some Fun with Translating from English into Logical Expressions

U = {fleegles, snurds, thingamabobs}

F(x): x is a fleegle

S(x): x is a snurd

T(x): x is a thingamabob

Translate: "Everything is a Fleegle."

Solution: $\forall x\, F(x)$

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Some Fun with Translating from English into Logical Expressions

U = {fleegles, snurds, thingamabobs}

F(x): x is a fleegle

S(x): x is a snurd

T(x): x is a thingamabob

"Nothing is a snurd."

Solution: $\neg\exists x\, S(x)$ What is this equivalent to?

Solution: $\forall x\, \neg S(x)$

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Some Fun with Translating from English into Logical Expressions

U = {fleegles, snurds, thingamabobs}

F(x): x is a fleegle

S(x): x is a snurd

T(x): x is a thingamabob

"All fleegles are snurds."

Solution: $\forall x\, (F(x) \rightarrow S(x))$

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Some Fun with Translating from English into Logical Expressions

U = {fleegles, snurds, thingamabobs}

F(x): x is a fleegle

S(x): x is a snurd

T(x): x is a thingamabob

"Some fleegles are thingamabobs."

Solution: $\exists x\, (F(x) \wedge T(x))$

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Some Fun with Translating from English into Logical Expressions

U = {fleegles, snurds, thingamabobs}

F(x): x is a fleegle

S(x): x is a snurd

T(x): x is a thingamabob

"No snurd is a thingamabob."

Solution: $\neg\exists x\, (S(x) \wedge T(x))$ What is this equivalent to?

Solution: $\forall x\, (\neg S(x) \vee \neg T(x))$

Solution: $\forall x\, (S(x) \rightarrow \neg T(x))$

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Some Fun with Translating from English into Logical Expressions

U = {fleegles, snurds, thingamabobs}

F(x): x is a fleegle

S(x): x is a snurd

T(x): x is a thingamabob

"If any fleegle is a snurd then it is also a thingamabob."

Solution: $\forall x\, ((F(x) \wedge S(x)) \rightarrow T(x))$

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System Specification Example

Predicate logic is used for specifying properties that systems must satisfy.

For example, translate into predicate logic:

- "Every mail message larger than one megabyte will be compressed."
- "If a user is active, at least one network link will be available."

Decide on predicates and domains (left implicit here) for the variables:

- Let L(m, y) be "Mail message m is larger than y megabytes."
- Let C(m) denote "Mail message m will be compressed."
- Let A(u) represent "User u is active."
- Let S(n, x) represent "Network link n is state x.

Now we have:

$\forall m\,(L(m, 1) \rightarrow C(m))$

$\exists u\, A(u) \rightarrow \exists n\, S(n, \text{available})$

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Lewis Carroll Example

Charles Lutwidge Dodgson (AKA Lewis Caroll) (1832-1898)

the first two are called premises and the third is called the conclusion

1. "All lions are fierce."
2. "Some lions do not drink coffee."
3. "Some fierce creatures do not drink coffee."

one way to translate these statements to predicate logic is to let P(x), Q(x), and R(x) be the propositional functions "x is a lion," "x is fierce," and "x drinks coffee," respectively.

1. $\forall x\,(P(x) \rightarrow Q(x))$
2. $\exists x\,(P(x) \land \neg R(x))$
3. $\exists x\,(Q(x) \land \neg R(x))$

Later we will see how to prove that the conclusion follows from the premises.

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Nested Quantifiers

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Section Summary

Nested Quantifiers

Order of Quantifiers

Translating from Nested Quantifiers into English

Translating Mathematical Statements into Statements involving Nested Quantifiers

Translated English Sentences into Logical Expressions

Negating Nested Quantifiers

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Nested Quantifiers

- nested quantifiers are often necessary to express the meaning of sentences in English as well as important concepts in computer science and mathematics
- **Example**: "Every real number has an inverse" is

  $\forall x\,\exists y\,(x + y = 0)$

  where the domains of x and y are the real numbers
- we can also think of nested propositional functions:

  $\forall x\,\exists y\,(x + y = 0)$ can be viewed as
  $\forall x\, Q(x)$ where Q(x) is
  $\exists y\, P(x, y)$ where
  P(x, y) is $(x + y = 0)$

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Thinking of Nested Quantification

nested loops

- to see if $\forall x\,\forall y\, P(x, y)$ is true, loop through the values of x:
  - at each step, loop through the values for y
  - if for some pair of x and y, P(x,y) is F, then $\forall x\,\forall y\, P(x, y)$ is F and both the outer and inner loops terminate

  $\forall x\,\forall y\, P(x, y)$ is T if the outer loop ends after stepping through each x

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Thinking of Nested Quantification

nested loops

- to see if ∀x ∃y P(x, y) is T, loop through the values of x:
  - at each step, loop through the values for y
  - the inner loop ends when a pair x and y is found such that P(x, y) is T
  - if no y is found such that P(x, y) is T the outer loop terminates as ∀x ∃y P(x, y) has been shown to be F

  ∀x ∃y P(x, y) is T if the outer loop ends after stepping through each x

if the domains of the variables are infinite, then this process cannot actually be carried out

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Order of Quantifiers

Examples:

1. let P(x,y): x + y = y + x and U is the real numbers

   ∀x ∀y P(x,y) and
   ∀y ∀x P(x,y) have the same truth value

2. let Q(x,y): x + y = 0 and U is the real numbers

   ∀x ∃y Q(x,y) is T
   ∃y ∀x Q(x,y) is F

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Questions on Order of Quantifiers

Example 1: Let U be the real numbers,

Define P(x,y) : x · y = 0

What is the truth value of the following:

1. ∀x ∀y P(x,y)
   False
2. ∀x ∃y P(x,y)
   True
3. ∃x ∀y P(x,y)
   True
4. ∃x ∃y P(x,y)
   True

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Questions on Order of Quantifiers

Example 2: Let U be the real numbers,

Define P(x,y) : x / y = 1

What is the truth value of the following:

1. ∀x ∀y P(x,y)
   False
2. ∀x ∃y P(x,y)
   False
3. ∃x ∀y P(x,y)
   False
4. ∃x ∃y P(x,y)
   True

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Quantifications of Two Variables

Statement	When True?	When False
$\forall x \forall y P(x,y)$ $\forall y \forall x P(x,y)$	*P(x,y)* is true for every pair *x,y*.	There is a pair *x, y* for which *P(x,y)* is false.
$\forall x \exists y P(x,y)$	For every *x* there is a *y* for which *P(x,y)* is true.	There is an x such that *P(x,y)* is false for every *y*.
$\exists x \forall y P(x,y)$	There is an *x* for which *P(x,y)* is true for every *y*.	For every *x* there is a y for which *P(x,y)* is false.
$\exists x \exists y P(x,y)$ $\exists y \exists x P(x,y)$	There is a pair *x, y* for which *P(x,y)* is true.	*P(x,y)* is false for every pair *x,y*

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Translating Nested Quantifiers into English

Example 1: Translate the statement

∀x (*C*(x) ∨ ∃y (*C*(y) ∧ F(x, y)))

where *C*(x) is "x has a computer," and F(x,y) is "x and y are friends," and the domain for both x and y consists of all students in your school

Solution: Every student in your school has a computer or has a friend who has a computer.

Example 2: Translate the statement

∃x∀y∀z ((F(x, y) ∧ F(x, z) ∧ (y ≠ z)) → ¬F(y, z))

Solution: There is a student, none of whose friends are also friends with each other.

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Translating Mathematical Statements into Predicate Logic

Example : Translate "The sum of two positive integers is always positive" into a logical expression.

Solution:

1. rewrite the statement to make the implied quantifiers and domains explicit:
   "For every two integers, if these integers are both positive, then the sum of these integers is positive."
2. introduce the variables x and y, and specify the domain, to obtain:
   "For all positive integers x and y, x + y is positive."
3. the result is:
   $\forall x\ \forall y\ (((x > 0) \wedge (y > 0)) \rightarrow (x + y > 0))$
   where the domain of both variables is all integers

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Translating English into Logical Expressions Example

Example: Use quantifiers to express the statement "There is a woman who has taken a flight on every airline in the world."

Solution:

1. let P(w,f) be "w has taken f" and Q(f,a) be "f is a flight on a"
2. the domain of w is all women, the domain of f is all flights, and the domain of a is all airlines
3. then the statement can be expressed as

$\exists w \forall a \exists f\ (P(w, f) \wedge Q(f, a))$

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Questions on Translation from English

Choose the obvious predicates and express in predicate logic.

Example 1: "Brothers are siblings."
Solution: $\forall x\ \forall y\ (B(x,y) \rightarrow S(x,y))$

Example 2: "Siblinghood is symmetric."
Solution: $\forall x\ \forall y\ (S(x,y) \rightarrow S(y,x))$

Example 3: "Everybody loves somebody."
Solution: $\forall x\ \exists y\ L(x,y)$

Example 4: "There is someone who is loved by everyone."
Solution: $\exists y\ \forall x\ L(x,y)$

Example 5: "There is someone who loves someone."
Solution: $\exists x\ \exists y\ L(x,y)$

Example 6: "Everyone loves himself"
Solution: $\forall x\ L(x,x)$

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Negating Nested Quantifiers

Example 1: Recall the logical expression developed three slides back:
$\exists w \forall a \exists f\ (P(w, f) \wedge Q(f, a))$

Part 1: Use quantifiers to express the statement that "There does not exist a woman who has taken a flight on every airline in the world."
Solution: $\neg\exists w \forall a \exists f\ (P(w, f) \wedge Q(f, a))$

Part 2: Now use De Morgan's Laws to move the negation as far inward as possible.
Solution:

1. $\neg\exists w \forall a \exists f\ (P(w, f) \wedge Q(f, a))$
2. $\forall w \neg\forall a \exists f\ (P(w, f) \wedge Q(f, a))$ DeMorgan's
3. $\forall w \exists a \neg\exists f\ (P(w, f) \wedge Q(f, a))$ DeMorgan's
4. $\forall w \exists a \forall f\ \neg(P(w, f) \wedge Q(f, a))$ DeMorgan's
5. $\forall w \exists a \forall f\ (\neg P(w, f) \vee \neg Q(f, a))$ DeMorgan's

Part 3: Can you translate the result back into English?
Solution:
"For every woman there is an airline such that for all flights, this woman has not taken that flight or that flight is not on this airline."

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Some Questions about Quantifiers (optional)

Can you switch the order of quantifiers?

- Is this a valid equivalence?
  $\forall x \forall y\ P(x, y) \equiv \forall y \forall x\ P(x, y)$
  - yes – the left and the right sides will always have the same truth value
  - the order in which x and y are picked does not matter
- Is this a valid equivalence?
  $\forall x \exists y\ P(x, y) \equiv \exists y \forall x\ P(x, y)$
  - no – the left and the right side may have different truth values for some propositional functions for P
    - try "x + y = 0" for P(x,y) with U being the integers
  - the order in which the values of x and y are picked does matter

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Some Questions about Quantifiers (optional)

Can you distribute quantifiers over logical connectives?

- Is this a valid equivalence?
  $\forall x\ (P(x) \wedge Q(x)) \equiv \forall x\ P(x) \wedge \forall x\ Q(x)$
  - yes – the left and the right side will always have the same truth value no matter what propositional functions are denoted by P(x) and Q(x)
- Is this a valid equivalence?
  $\forall x\ (P(x) \rightarrow Q(x)) \equiv \forall x\ P(x) \rightarrow \forall x\ Q(x)$
  - no; let P(x) = "x is a fish" and Q(x) = "x has scales" with the domain of discourse being all animals
  - then the left side is false, because there are some fish that do not have scales, but the right side is true since not all animals are fish

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