The Foundations: Logic and Proofs

Chapter 1, Part I: Propositional Logic

Chapter Summary

Propositional Logic

- The Language of Propositions
- Applications
- Logical Equivalences

Predicate Logic

- The Language of Quantifiers
- Logical Equivalences
- Nested Quantifiers

Proofs

- Rules of Inference
- Proof Methods
- Proof Strategy

Propositional Logic Summary

The Language of Propositions

- Connectives
- Truth Values
- Truth Tables

Applications

- Translating English Sentences
- System Specifications
- Logic Puzzles
- Logic Circuits

Logical Equivalences

- Important Equivalences
- Showing Equivalence
- Satisfiability

Propositional Logic

Section 1.1

Propositions

- proposition: a statement that is either true or false
- examples
  - Today is sunny.
  - Trenton is the capital of New Jersey.
  - Toronto is the capital of Canada.
  - $1 + 0 = 1$
  - $0 + 0 = 2$
- examples that are not propositions
  - Sit down!
  - What time is it?
  - $x + 1 = 2$
  - $x + y = z$

Constructing Propositions

- propositional variables: p, q, r, s, …
- the proposition that is always true is denoted by **T**
- the proposition that is always false is denoted by **F**
- compound propositions: constructed from logical connectives and other propositions
  - negation $\neg$
  - conjunction $\wedge$
  - disjunction $\vee$
  - implication $\rightarrow$
  - biconditional $\leftrightarrow$

Compound Propositions: Negation

- negation of a proposition p denoted by $\neg p$

p	$\neg p$
T	F
F	T

- example

  p: "The earth is round."

  $\neg p$: "It is not the case that the earth is round."

  or more simply "The earth is not round."

Conjunction (and)

- conjunction of p and q: $p \wedge q$

p	q	$p \wedge q$
T	T	T
T	F	F
F	T	F
F	F	F

- example
  p: "I am at home."
  q: "It is raining."
  $p \wedge q$: "I am at home and it is raining."

Disjunction (or)

- disjunction of p and q: $p \vee q$

p	q	$p \vee q$
T	T	T
T	F	T
F	T	T
F	F	F

- example
  p: "I am at home."
  q: "It is raining."
  $p \vee q$: "I am at home or it is raining."

Exclusive or

- in English "or" has two distinct meanings:
  - inclusive or $(p \lor q)$
    - example: "Do you want cream or sugar with coffee?"
    - customers may want one, the other, or both
    - if $p \lor q$ is T, either one or both of p and q must be T
  - exclusive or $(p \oplus q)$
    - example: "Soup or salad comes with this entrée"
    - we do not expect to be able to get both soup and salad
    - one of p and q must be true, but not both

Exclusive or

- $p \oplus q$

p	q	$p \oplus q$
T	T	F
T	F	T
F	T	T
F	F	F

Implication (if then)

- conditional statement or implication: $p \rightarrow q$
  - read: if p, then q
  - p is the hypothesis (antecedent or premise)
  - q is the conclusion (or consequence)

p	q	$p \rightarrow q$
T	T	T
T	F	F
F	T	T
F	F	T

- example:
  p: "I am at home." q: "It is raining."
  $p \rightarrow q$: "If I am at home then it is raining."

Understanding Implication

- in $p \rightarrow q$ no need for any connection between the antecedent or the consequent
- the meaning of $p \rightarrow q$ depends only on the truth values of p and q
- the following implications are perfectly fine, but would not be used in ordinary English
  - If my car gets less than 30 mpg, then I have more money than Elon Musk.
  - If tomorrow is July 4th, then I need my winter coat.
  - If $1 + 1 = 3$, then my grandma got hit by a parked car.

Understanding Implication

- one way to view the logical conditional is to think of an obligation or contract
  - If I am elected, then I will lower taxes.
  - If you get 100% on the final, then you will get an *A*.
- If the politician is elected and does not lower taxes, then the voters can say that he or she has broken the campaign pledge. Something similar holds for the professor. This corresponds to the case where p is true and q is false.

Different Ways of Stating $p \rightarrow q$

if p, then q

if p, q

q unless ¬p

q if p

q whenever p

q follows from p

p implies q

p only if q

q when p

p is sufficient for q

q is necessary for p

a necessary condition for p is q

a sufficient condition for q is p

Converse, Contrapositive, Inverse

- from $p \rightarrow q$ we can form new conditional statements
  - $q \rightarrow p$ converse of $p \rightarrow q$
  - $\neg q \rightarrow \neg p$ contrapositive of $p \rightarrow q$
  - $\neg p \rightarrow \neg q$ inverse of $p \rightarrow q$

- example: find the converse, contrapositive, and inverse of "Raining is a sufficient condition for my not going to town."
  - translation: If it is raining, then I do not go to town.
  - converse: If I do not go to town, then it is raining.
  - contrapositive: If I go to town, then it is not raining.
  - inverse: If it is not raining, then I will go to town.

Biconditional (iff)

- biconditional: $p \leftrightarrow q$
  - read as "p if and only if q"

p	q	$p \leftrightarrow q$
T	T	T
T	F	F
F	T	F
F	F	T

- example:
  p: "I am at home."
  q: "It is raining."
  $p \leftrightarrow q$: "I am at home if and only if it is raining."

Expressing the Biconditional

- alternative ways "p if and only if q" is expressed in English
  - p is necessary and sufficient for q
  - p iff q

Truth Tables For Compound Propositions

- construction of a truth table
  - rows
    - need a row for every possible combination of values
    - state them in order
  - columns
    - need a column for the compound proposition (usually at far right)
    - need a column for each expression that occurs in the compound proposition as it is built up
    - including atomic propositions

Example Truth Table

- construct a truth table for $p \vee q \rightarrow \neg r$

p	q	r	$\neg r$	$p \vee q$	$p \vee q \rightarrow \neg r$
T	T	T	F	T	F
T	T	F	T	T	T
T	F	T	F	T	F
T	F	F	T	T	T
F	T	T	F	T	F
F	T	F	T	T	T
F	F	T	F	F	T
F	F	F	T	F	T

Equivalent Propositions

- two propositions are equivalent if they always have the same truth value.
- example: use a truth table to show that the conditional $(p \rightarrow q)$ is equivalent to the contrapositive $(\neg q \rightarrow \neg p)$
- solution:

p	q	$\neg p$	$\neg q$	$p \rightarrow q$	$\neg q \rightarrow \neg p$
T	T	F	F	T	T
T	F	F	T	F	F
F	T	T	F	T	T
F	F	T	T	T	T

Using a Truth Table to Show Non-Equivalence

- example: Show using a truth table that neither the converse ($q \rightarrow p$) nor inverse ($\neg p \rightarrow \neg q$) of an implication are equivalent to the implication ($p \rightarrow q$).
- solution:

p	q	$\neg p$	$\neg q$	$p \rightarrow q$	$\neg p \rightarrow \neg q$	$q \rightarrow p$
T	T	F	F	T	T	T
T	F	F	T	F	T	T
F	T	T	F	T	F	F
F	F	T	T	T	T	T

Problem

- how many rows are there in a truth table with n propositional variables?
- answer: 2^n
- note that this means that with n propositional variables, we can construct 2^n distinct (that is, not equivalent) propositions

Precedence of Logical Operators

Operator	Precedence
$\neg$	1
$\wedge$	2
$\vee$	3
$\rightarrow$	4
$\leftrightarrow$	5

- $p \vee q \rightarrow \neg r$ is equivalent to $(p \vee q) \rightarrow \neg r$
- if the intended meaning is $p \vee (q \rightarrow \neg r)$ then parentheses must be used

Applications of Propositional Logic

Section 1.2

Applications of Propositional Logic: Summary

- Translating English to Propositional Logic
- System Specifications
- Boolean Searching
- Logic Puzzles
- Logic Circuits
- AI Diagnosis Method (Optional)

Translating English Sentences

- steps to convert an English sentence to a statement in propositional logic
  - identify atomic propositions and represent using propositional variables in the affirmative
  - determine appropriate logical connectives
- example: "If I go to Harry's or to the country, I will not go shopping."
  - p: I go to Harry's
  - q: I go to the country
  - r: I will go shopping

if p or q then not r

$$(p \vee q) \rightarrow \neg r$$

Example

- Problem: Translate the following sentence into propositional logic:

  "You can access the Internet from campus only if you are a computer science major or you are not a freshman."

- One Solution: Let a, c, and f represent

  a: You can access the internet from campus
  c: You are a computer science major
  f: You are a freshman

  restate: If you can access the internet, then you are a computer science major or you are not a freshman.

$$a \rightarrow (c \lor \neg f)$$

System Specifications

- take system requirements in English and express them as precise logic statements
- example: "An automated reply cannot be sent when the file system is full."
- solution:

p: "The automated reply can be sent."
q: "The file system is full."

restate: "If the file system is full, an automated reply cannot be sent."

$$q \rightarrow \neg p$$

Consistent System Specifications

- Definition: A list of propositions is consistent if it is possible to assign truth values to the proposition variables so that each proposition is true.
- example: Are these specifications consistent?
  - The diagnostic message is stored in the buffer or it is retransmitted.
  - The diagnostic message is not stored in the buffer.
  - If the diagnostic message is stored in the buffer, then it is retransmitted.
- solution

  p: The diagnostic message is stored in the buffer.
  q: The diagnostic message is retransmitted.

the specifications can be written as

$p \vee q$
$\neg p$
$p \rightarrow q$

when p is false and q is true all three statements are true, so the specification is consistent

Consistent System Specifications

- (continued)
  p: The diagnostic message is stored in the buffer.
  q: The diagnostic message is retransmitted.

  the specifications can be written as
  $p \vee q$
  $\neg p$
  $p \rightarrow q$

  What if the following specification is added?

  The diagnostic message is not retransmitted

  - solution: Now we are adding $\neg q$ and there is no satisfying assignment. So, the specification is not consistent.

Logic Puzzles

Raymond Smullyan (Born 1919)

An island has two kinds of inhabitants:
knights, who always tell the truth
knaves, who always lie

You go to the island and meet A and B.

- A says "B is a knight."
- B says "The two of us are of opposite types."

Problem: What are the types of A and B?

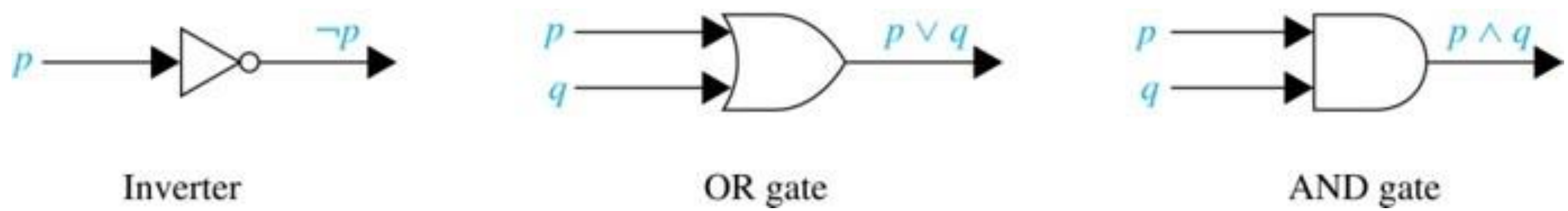
Solution: Let
p: A is a knight
q: B is a knight

If A is a knight, then p is true. Since knights tell the truth, q must also be true, so $(p \wedge q)$. Then by B's statement, $(p \wedge \neg q) \vee (\neg p \wedge q)$ would have to be true, but it is not. So, A is not a knight and therefore $\neg p$ must be true.

If A is a knave, then B must not be a knight since knaves always lie. So, then both $\neg p$ and $\neg q$ hold since both are knaves: $\neg p \wedge \neg q$.

Logic Circuits

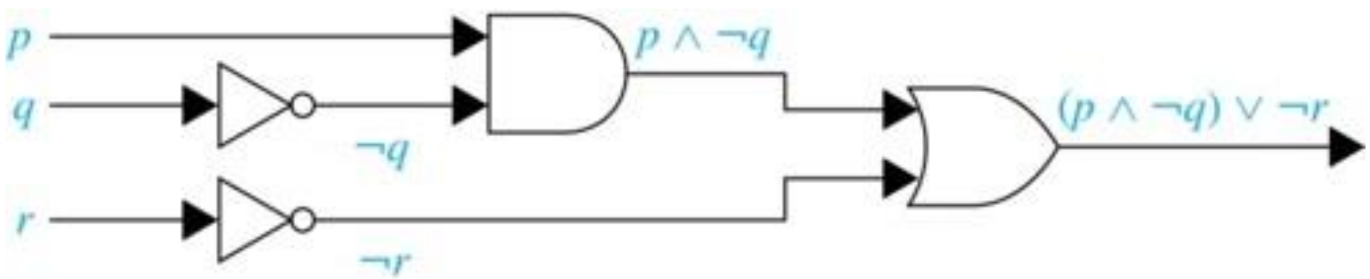
- in electronic circuits, each input/output signal can be viewed as a 0 or 1
  - 0 represents **False**
  - 1 represents **True**
- complicated circuits are constructed from three basic circuits called gates



  - inverter (NOT gate): negates input bit
  - OR gate: takes disjunction of the two input bits
  - AND gate: takes conjunction of the two input bits

Logic Circuits

- more complicated digital circuits can be constructed by combining these basic circuits to produce a desired output given the input signals by building a circuit for each piece of the output expression and then combining them.
- example



Propositional Equivalences

Section 1.3

Section Summary

- Tautologies, Contradictions, and Contingencies
- Logical Equivalence
  - Important Logical Equivalences
  - Showing Logical Equivalence
- Normal Forms
  - Disjunctive Normal Form
  - Conjunctive Normal Form
- Propositional Satisfiability
  - Sudoku Example

Tautologies, Contradictions, and Contingencies

- tautology: a proposition which is always true
  - example: $p \vee \neg p$
- contradiction: a proposition which is always false
  - example: $p \wedge \neg p$
- contingency: a proposition which is neither a tautology nor a contradiction, such as p

p	$\neg p$	$p \vee \neg p$	$p \wedge \neg p$
T	F	T	F
F	T	T	F

Logically Equivalent

- two compound propositions p and q are logically equivalent if and only if the columns in a truth table giving their truth values agree
- can also be stated as two compound propositions p and q are logically equivalent if $p \leftrightarrow q$ is a tautology
  - denoted as $p \equiv q$ or $p \Leftrightarrow q$
  - this truth table shows that $\neg p \vee q$ is equivalent to $p \rightarrow q$

p	q	$\neg p$	$\neg p \vee q$	$p \rightarrow q$	$(\neg p \vee q) \leftrightarrow (p \rightarrow q)$
T	T	F	T	T	T
T	F	F	F	F	T
F	T	T	T	T	T
F	F	T	T	T	T

De Morgan's Laws

$\neg(p \wedge q) \equiv \neg p \vee \neg q$
$\neg(p \vee q) \equiv \neg p \wedge \neg q$

Augustus De Morgan 1806-1871

truth table showing De Morgan's Second Law is true

p	q	$\neg p$	$\neg q$	$(p \vee q)$	$\neg(p \vee q)$	$\neg p \wedge \neg q$
T	T	F	F	T	F	F
T	F	F	T	T	F	F
F	T	T	F	T	F	F
F	F	T	T	F	T	T

Key Logical Equivalences

Identity Laws:	$p \wedge T \equiv p$	$p \vee F \equiv p$
Domination Laws:	$p \vee T \equiv T$	$p \wedge F \equiv F$
Idempotent laws:	$p \vee p \equiv p$	$p \wedge p \equiv p$
Double Negation Law:	$\neg(\neg p) \equiv p$	
Negation Laws:	$p \vee \neg p \equiv T$	$p \wedge \neg p \equiv F$

Key Logical Equivalences

Commutative Laws: $p \lor q \equiv q \lor p$ $\quad p \land q \equiv q \land p$

Associative Laws: $(p \land q) \land r \equiv p \land (q \land r)$

$(p \lor q) \lor r \equiv p \lor (q \lor r)$

Distributive Laws: $p \lor (q \land r) \equiv (p \lor q) \land (p \lor r)$

$p \land (q \lor r) \equiv (p \land q) \lor (p \land r)$

Absorption Laws: $p \lor (p \land q) \equiv p$ $\quad p \land (p \lor q) \equiv p$

More Logical Equivalences

Implication Rule: $p \rightarrow q \equiv \neg p \vee q$

$p \vee q \equiv \neg p \rightarrow q$

Contrapositive: $p \rightarrow q \equiv \neg q \rightarrow \neg p$

Biconditional: $p \leftrightarrow q \equiv (p \rightarrow q) \wedge (q \rightarrow p)$

Contrapositive of Biconditional: $p \leftrightarrow q \equiv \neg p \leftrightarrow \neg q$

Constructing New Logical Equivalences

- show two expressions are logically equivalent by developing a series of logically equivalent statements
- to prove that $A \equiv B$ we produce a series of equivalences beginning with A and ending with B

$$A \equiv A_1$$

$$\ldots$$

$$A_n \equiv B$$

- a proposition (represented by a propositional variable) may be replaced by an arbitrarily complex compound proposition
- only one replacement per line allowed (except assoc/ comm)

Equivalence Proofs

Example: Show that $\neg(p \vee (\neg p \wedge q)) \equiv \neg p \wedge \neg q$

Solution:

$\neg(p \vee (\neg p \wedge q)) \equiv \neg p \wedge \neg(\neg p \wedge q)$	De Morgan's law
$\equiv \neg p \wedge [\neg(\neg p) \vee \neg q]$	De Morgan's law
$\equiv \neg p \wedge (p \vee \neg q)$	double negation
$\equiv (\neg p \wedge p) \vee (\neg p \wedge \neg q)$	distributive
$\equiv F \vee (\neg p \wedge \neg q)$	negation
$\equiv (\neg p \wedge \neg q) \vee F$	commutative
$\equiv \neg p \wedge \neg q$	identity

Equivalence Proofs

Example: Show that $(p \wedge q) \rightarrow (p \vee q)$ is a tautology

Solution:

$(p \wedge q) \rightarrow (p \vee q)$	$\equiv \neg(p \wedge q) \vee (p \vee q)$	implication rule
	$\equiv (\neg p \vee \neg q) \vee (p \vee q)$	De Morgan's law
	$\equiv (\neg p \vee p) \vee (\neg q \vee q)$	associative and commutative
	$\equiv T \vee (\neg q \vee q)$	negation
	$\equiv T \vee T$	negation
	$\equiv T$	domination

Propositional Satisfiability

- a compound proposition is satisfiable if there is an assignment of truth values to its variables that make it true
- when no such assignments exist, the compound proposition is unsatisfiable
- a compound proposition is unsatisfiable if and only if its negation is a tautology

Questions on Propositional Satisfiability

- example: determine the satisfiability of the following compound propositions:

$$(p \vee \neg q) \wedge (q \vee \neg r) \wedge (r \vee \neg p)$$

solution: satisfiable – assign T to p, q, and r

$$(p \vee q \vee r) \wedge (\neg p \vee \neg q \vee \neg r)$$

solution: satisfiable – assign T to p and F to q

$$(p \vee \neg q) \wedge (q \vee \neg r) \wedge (r \vee \neg p) \wedge (p \vee q \vee r) \wedge (\neg p \vee \neg q \vee \neg r)$$

solution: not satisfiable – no assignment of truth values to the propositional variables will make the proposition true

Notation

$$\bigvee_{j=1}^{n} p_j \text{ is used for } p_1 \vee p_2 \vee \ldots \vee p_n$$

$$\bigwedge_{j=1}^{n} p_j \text{ is used for } p_1 \wedge p_2 \wedge \ldots \wedge p_n$$

Needed for the next example.

Sudoku

- a Sudoku puzzle is represented by a 9×9 grid made up of nine 3×3 subgrids, known as blocks
- some of the 81 cells of the puzzle are assigned one of the numbers 1,2, …, 9
- the puzzle is solved by assigning numbers to each blank cell so that every row, column and block contains each of the nine possible numbers.
- example:

	2	9				4		
			5			1		
	4							
				4	2			
6							7	
5								
7			3					5
	1			9				
							6	

Encoding as a Satisfiability Problem

- let p(i,j,n) denote the proposition that is true when the number n is in the cell in the i^{th} row and the j^{th} column
- there are $9 \times 9 \times 9 = 729$ such propositions
- in the sample puzzle p(5,1,6) is true, but p(5,j,6) is false for j = 2,3,…9

	2	9				4		
			5			1		
	4							
				4	2			
6							7	
5								
7			3					5
	1			9				
							6	

Encoding as a Satisfiability Problem

- for each cell with a given value, assert p(i,j,n), when the cell in row i and column j has the given value
- assert that every row contains every number

$$\bigwedge_{i=1}^{9}\bigwedge_{n=1}^{9}\bigvee_{j=1}^{9} p(i,j,n)$$

- assert that every column contains every number

$$\bigwedge_{j=1}^{9}\bigwedge_{n=1}^{9}\bigvee_{i=1}^{9} p(i,j,n)$$

Encoding as a Satisfiability Problem

- assert that each of the 3 × 3 blocks contain every number

$$\bigwedge_{r=0}^{2}\bigwedge_{s=0}^{2}\bigwedge_{n=1}^{9}\bigwedge_{i=1}^{3}\bigvee_{j=1}^{3} p(3r+i,3s+j,n)$$

- assert that no cell contains more than one number
  - take the conjunction over all values of n, n′, i, and j, where each variable ranges from 1 to 9 and n ≠ n′, of

$$p(i,j,n) \to \neg p(i,j,n')$$

Solving Satisfiability Problems

- to solve a Sudoku puzzle, we need to find an assignment of truth values to the 729 variables of the form $p(i,j,n)$ that makes the conjunction of the assertions true; those variables that are assigned T yield a solution to the puzzle
- a truth table can always be used to determine the satisfiability of a compound proposition, but this is too complex even for modern computers for large problems
- there has been much work on developing efficient methods for solving satisfiability problems as many practical problems can be translated into satisfiability problems