Basic Structures: Sets, Functions, Sequences, Sums, and Matrices

Chapter 2

With Question/Answer Animations

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Chapter Summary

- Sets
  - The Language of Sets
  - Set Operations
  - Set Identities
- Functions
  - Types of Functions
  - Operations on Functions
  - Computability
- Sequences and Summations
  - Types of Sequences
  - Summation Formulae
- Set Cardinality
  - Countable Sets
- Matrices
  - Matrix Arithmetic

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Sets

Section 2.1

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Section Summary

- Definition of sets
- Describing Sets
  - Roster Method
  - Set-Builder Notation
- Some Important Sets in Mathematics
- Empty Set and Universal Set
- Subsets and Set Equality
- Cardinality of Sets
- Tuples
- Cartesian Product

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Introduction

- sets: basic building blocks for objects in discrete mathematics
  - important for counting
  - programming languages have set operations
- set theory: important branch of mathematics
  - many different systems of axioms have been used to develop set theory
  - we are not concerned with a formal set of axioms for set theory
  - instead, we will use naïve set theory

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Sets

- set: an unordered collection of objects
  - the students in this class
  - the chairs in this room
- objects in a set are called elements, or members, of the set; a set contains its elements
- the notation $a \in A$ denotes that a is an element of set A
- if a is not a member of A, write $a \notin A$

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Describing a Set: Roster Method

- S = {a, b, c, d}
  - order not important
  - S = {a, b, c, d} = {b, c, a, d}
- each distinct object is either a member or not; listing more than once does not change the set.
  - S = {a, b, c, d} = {a, b, c, b, c, d}
- ellipses (…) may be used to describe a set without listing all the members when the pattern is clear
  - S = {a, b, c, d, …, z}

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Roster Method Examples

- set of all vowels in the English alphabet
  - V = {a, e, i, o, u}
- set of all odd positive integers less than 10
  - O = {1, 3, 5, 7, 9}
- set of all positive integers less than 100
  - S = {1, 2, 3, …, 99}
- set of all integers less than 0
  - S = {…, -3, -2, -1}

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Some Important Sets

- **N** = natural numbers = {1, 2, 3, …}
- **W** = whole numbers = {0, 1, 2, 3, …}
- **Z** = integers = {…, -3, -2, -1, 0, 1, 2, 3, …}
- $\mathbf{Z}^+$ = positive integers = {1, 2, 3,…}
- **R** = set of real numbers
- $\mathbf{R}^+$ = set of positive real numbers
- ***C*** = set of complex numbers
- **Q** = set of rational numbers

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Set-Builder Notation

- specify the property that all members must satisfy
  - S = {x | x is a positive integer less than 100}
  - O = {x | x is an odd positive integer less than 10}
  - O = {x ∈ $\mathbf{Z}^+$ | x is odd and x < 10}
- a predicate may be used
  - S = {x | P(x)}
  - example: S = {x | Prime(x)}
- positive rational numbers
  - $\mathbf{Q}^+$ = {x ∈ **R** | x = p/q, for some positive integers p,q}

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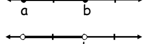
Interval Notation

- [a, b] = {x | a ≤ x ≤ b}
- [a, b) = {x | a ≤ x < b}
- (a, b] = {x | a < x ≤ b}
- (a, b) = {x | a < x < b}

- closed interval [a, b]
- open interval (a, b) 

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Universal Set and Empty Set

- universal set U
  - set containing everything currently under consideration
  - sometimes implicit
  - sometimes explicitly stated
  - contents depend on the context
- empty set: set with no elements
  - symbolized with ∅ or { }

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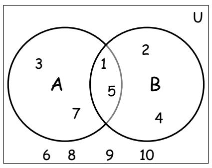
Venn Diagrams

- outer box represents universal set U
  - all elements not in any set should be listed here for reasonable finite sets
- inner circles represent individual sets
- example: $A = \{1, 3, 5, 7\}$ $B = \{1, 2, 4, 5\}$ $U = \{1, 2, \ldots, 10\}$



John Venn
(1834-1923)
Cambridge, UK

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Russell's Paradox

- let *S* be the set of all sets which are not members of themselves; a paradox results from trying to answer the question "is *S* a member of itself?"
- related paradox
  - Henry is a barber who shaves all men who do not shave themselves
  - a paradox results from trying to answer the question "Does Henry shave himself?"

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Some Things to Remember

- sets can be elements of sets
  - $\{\{1, 2, 3\}, a, \{b, c\}\}$
  - $\{\mathbf{N}, \mathbf{Z}, \mathbf{Q}, \mathbf{R}\}$
- the empty set is different from a set containing the empty set
  - $\emptyset \neq \{\emptyset\}$

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Set Equality

- **Definition**: Two sets are equal if and only if they have the same elements.
  - therefore, sets A and B are equal if and only if

    $\forall x\,(x \in A \leftrightarrow x \in B)$
  - we write $A = B$ if A and B are equal sets
    - $\{1, 3, 5\} = \{3, 5, 1\}$
    - $\{1, 5, 5, 5, 3, 3, 1\} = \{1, 3, 5\}$

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Subsets

Definition: Set A is a subset of B if and only if every element of A is also an element of B.

- the notation $A \subseteq B$ is used to indicate that A is a subset of the set B
- $A \subseteq B$ holds if and only if $\forall x\,(x \in A \rightarrow x \in B)$ is true
  - since $a \in \emptyset$ is always false, $\emptyset \subseteq S$ for every set S
  - since $(a \in S) \rightarrow (a \in S)$, $S \subseteq S$ for every set S

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Showing a Set Is or Is Not a Subset of Another Set

- show $A \subseteq B$
  - show that if x belongs to A, then x also belongs to B
- show $A \not\subseteq B$
  - find an element $x \in A$ and $x \notin B$
  - x is a counterexample to the claim that $x \in A \rightarrow x \in B$
- examples
  - the set of all computer science majors at your school is a subset of all students at your school
  - the set of integers with squares less than 100 is not a subset of the set of nonnegative integers

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Another Look at Equality of Sets

- $A = B$ if and only if

  $\forall x\,(x \in A \leftrightarrow x \in B)$
- using logical equivalences, $A = B$ iff

  $\forall x\,[(x \in A \rightarrow x \in B) \wedge (x \in B \rightarrow x \in A)]$
- which is equivalent to

  $A \subseteq B$ and $B \subseteq A$

  or $A = B$

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Proper Subsets

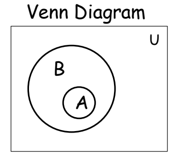
Definition: If $A \subseteq B$, but $A \neq B$, then A is a proper subset of B, denoted by $A \subset B$. If $A \subset B$, then

$$\forall x\,(x \in A \rightarrow x \in B) \wedge \exists x\,(x \in B \wedge x \notin A)$$

is true

- it's a subset, just not the whole set

Venn Diagram



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Set Cardinality

- if there are exactly n distinct elements in S where n is a nonnegative integer, S is finite
  - otherwise, it is infinite
- the cardinality of a finite set A, denoted by $|A|$, is the number of (distinct) elements of A
- examples
  1. $|\emptyset| = 0$
  2. let S: letters of the English alphabet; then $|S| = 26$
  3. $|\{1, 2, 3\}| = 3$
  4. $|\{\emptyset\}| = 1$
  5. the set of integers is infinite

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Power Sets

- power set: the set of all subsets of a set A
  - denoted $P(A)$
  - example: $A = \{a, b\}$ then

    $P(A) = \{\emptyset, \{a\}, \{b\}, \{a, b\}\}$
  - if a set has n elements, then the cardinality of the power set is 2^n

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Tuples

- an ordered n-tuple $(a_1, a_2, \ldots, a_n)$ is an ordered collection that has a_1 as its first element, a_2 as its second element, and so on until a_n as its last element
- two n-tuples are equal if and only if their corresponding elements are equal
- 2-tuples are called ordered pairs
  - ordered pairs (a, b) and (c, d) are equal if and only if $a = c$ and $b = d$

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Cartesian Product

René Descartes (1596-1650)

- the Cartesian Product of two sets A and B, denoted by $A \times B$ is the set of ordered pairs (a, b) where $a \in A$ and $b \in B$

  $A \times B = \{(a, b) \mid a \in A \wedge b \in B\}$
- example

  $A = \{a,b\}$ $B = \{1,2,3\}$

  $A \times B = \{(a, 1), (a, 2), (a, 3), (b, 1), (b, 2), (b, 3)\}$
- a subset R of the Cartesian product $A \times B$ is called a relation from the set A to the set B (Chapter 9)

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Cartesian Product

- the Cartesian products of the sets $A_1, A_2, \ldots\ldots, A_n$, denoted by $A_1 \times A_2 \times \ldots\ldots \times A_n$, is the set of ordered n-tuples $(a_1, a_2, \ldots, a_n)$ where a_i belongs to A_i for i = 1, …, n

$$A_1 \times A_2 \times \cdots \times A_n = \{(a_1, a_2 \cdots, a_n) \mid a_i \in A_i \text{ for } i = 1, 2, \ldots n\}$$

- example: what is A × B × C where

  A = {0, 1}, B = {1, 2} and C = {0, 1, 2}

- solution: A × B × C = {(0, 1, 0), (0, 1, 1), (0, 1, 2), (0, 2, 0), (0, 2, 1), (0, 2, 2), (1, 1, 0), (1, 1, 1), (1, 1, 2), (1, 2, 0), (1, 2, 1), (1, 2, 2)}

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Truth Sets of Quantifiers

- Given a predicate P and a domain D, we define the truth set of P to be the set of elements in D for which P(x) is true
  - the truth set of P(x) is denoted by

    {x ∈ D | P(x)}

  - example: the truth set of P(x) where the domain is the integers and P(x): |x| = 1 is the set {-1, 1}

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Set Operations

Section 2.2

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Section Summary

- Set Operations
  - Union
  - Intersection
  - Complementation
  - Difference
- More on Set Cardinality
- Set Identities
- Proving Identities
- Membership Tables

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Boolean Algebra

- propositional calculus and set theory are both instances of an algebraic system called Boolean Algebra
- the operators in set theory are analogous to the corresponding operator in propositional calculus
- as always there must be a universal set U; all sets are assumed to be subsets of U

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Union

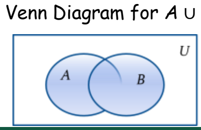
Definition: Let A and B be sets. The union of the sets A and B, denoted by A ∪ B, is the set:

{x | x ∈ A ∨ x ∈ B}

Example: What is {1, 2, 3} ∪ {3, 4, 5}?

Solution: {1, 2, 3, 4, 5}

Venn Diagram for A ∪ B



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Intersection

Definition: The intersection of sets A and B, denoted by A ∩ B, is

$$\{x \mid x \in A \wedge x \in B\}$$

Note if the intersection is empty, then A and B are said to be disjoint.

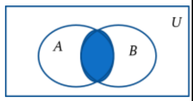
Example: What is? {1, 2, 3} ∩ {3, 4, 5} ?

Solution: {3}

Example: What is {1, 2, 3} ∩ {4, 5, 6} ?

Solution: ∅

Venn Diagram for A ∩ B



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Complement

Definition: If A is a set, then the complement of the A (with respect to U), denoted by $\bar{A}$, is the set U – A

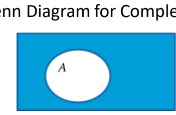
$$\bar{A} = \{x \mid x \in U \wedge x \notin A\}$$

- $\bar{A}$ is sometimes denoted A^c

Example: If U is the positive integers less than 100, what is the complement of {x | x > 70}

Solution: {x | x ≤ 70}

Venn Diagram for Complement



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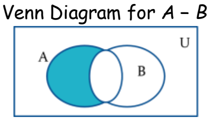
Difference

Definition: The difference of sets A and B, denoted by A – B, is the set containing the elements of A that are not in B.

$$A - B = \{x \mid x \in A \wedge x \notin B\} = A \cap \bar{B}$$

- A – B is also called the complement of B with respect to A

Venn Diagram for A – B



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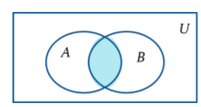
The Cardinality of the Union of Two Sets

Inclusion-Exclusion

$$|A \cup B| = |A| + |B| - |A \cap B|$$

Example: Let A be the math majors in your class and B be the CS majors. To count the number of students who are either math majors or CS majors, add the number of math majors and the number of CS majors, and subtract the number of joint CS/math majors.

Venn Diagram for A, B, A ∩ B, A ∪ B



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Review Questions

- **Example**: U = {0, 1, 2, 3, 4, 5, 6, 7, 8, 9, 10}
  A = {1, 2, 3, 4, 5}, B = {4, 5, 6, 7, 8}

1. A ∪ B

   Solution: {1, 2, 3, 4, 5, 6, 7, 8}
2. A ∩ B

   Solution: {4, 5}
3. $\bar{A}$

   Solution: {0, 6, 7, 8, 9, 10}
4. $\bar{B}$

   Solution: {0, 1, 2, 3, 9, 10}
5. A – B

   Solution: {1, 2, 3}
6. B – A

   Solution: {6, 7, 8}

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Symmetric Difference

Definition: The symmetric difference of **A** and **B**, denoted by $A \oplus B$ is the set

$$(A - B) \cup (B - A)$$

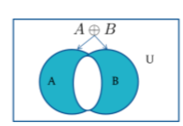
Example:

U = {0, 1, 2, 3, 4, 5, 6, 7, 8, 9, 10}

A = {1, 2, 3, 4, 5} B ={4, 5, 6, 7, 8}

What is $A \oplus B$:

Solution: {1, 2, 3, 6, 7, 8}



Venn Diagram

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Set Identities

Identity laws

$A \cup \varnothing = A \qquad A \cap U = A$

Domination laws

$A \cup U = U \qquad A \cap \varnothing = \varnothing$

Idempotent laws

$A \cup A = A \qquad A \cap A = A$

Complementation law

$\left(\overline{\overline{A}}\right) = A$

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Set Identities

Commutative laws

$A \cup B = B \cup A \qquad A \cap B = B \cap A$

Associative laws

$A \cup (B \cup C) = (A \cup B) \cup C$

$A \cap (B \cap C) = (A \cap B) \cap C$

Distributive laws

$A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$

$A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$

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Set Identities

De Morgan's laws

$\overline{A \cup B} = \overline{A} \cap \overline{B} \qquad \overline{A \cap B} = \overline{A} \cup \overline{B}$

Absorption laws

$A \cup (A \cap B) = A \qquad A \cap (A \cup B) = A$

Complement laws

$A \cup \overline{A} = U \qquad A \cap \overline{A} = \varnothing$

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Proving Set Identities

- Different ways to prove set identities:
  - Prove that each set (side of the identity) is a subset of the other.
  - Use set builder notation and propositional logic.
  - Membership Tables: Verify that elements in the same combination of sets always either belong or do not belong to the same side of the identity. Use 1 to indicate it is in the set and a 0 to indicate that it is not

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Proof of Second De Morgan Law

- **Example**: Prove that $\overline{A \cap B} = \overline{A} \cup \overline{B}$
- **Solution**: We prove this identity by showing that:

1) $\overline{A \cap B} \subseteq \overline{A} \cup \overline{B}$ and

2) $\overline{A} \cup \overline{B} \subseteq \overline{A \cap B}$

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Proof of Second De Morgan Law

- These steps show that: $\overline{A \cap B} \subseteq \overline{A} \cup \overline{B}$

$x \in \overline{A \cap B}$	by assumption
$x \notin A \cap B$	defn. of complement
$\neg((x \in A) \wedge (x \in B))$	by defn. of intersection
$\neg(x \in A) \vee \neg(x \in B)$	1st De Morgan law for Prop Logic
$x \notin A \vee x \notin B$	defn. of negation
$x \in \overline{A} \vee x \in \overline{B}$	defn. of complement
$x \in \overline{A} \cup \overline{B}$	by defn. of union

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Proof of Second De Morgan Law

- These steps show that: $\overline{A} \cup \overline{B} \subseteq \overline{A \cap B}$

$x \in \overline{A} \cup \overline{B}$	by assumption
$(x \in \overline{A}) \vee (x \in \overline{B})$	by defn. of union
$(x \notin A) \vee (x \in \overline{B})$	defn. of complement
$\neg(x \in A) \vee \neg(x \in B)$	defn. of negation
$\neg((x \in A) \wedge \neg(x \in B))$	1st De Morgan law for Prop Logic
$\neg(x \in A \cap B)$	defn. of intersection
$x \in \overline{A \cap B}$	defn. of complement

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Set-Builder Notation: Second De Morgan Law

$\overline{A \cap B} = x \in \overline{A \cap B}$	by defn. of complement
$= \{x \mid \neg(x \in (A \cap B))\}$	by defn. of does not belong symbol
$= \{x \mid \neg(x \in A \wedge x \in B\}$	by defn. of intersection
$= \{x \mid \neg(x \in A) \vee \neg(x \in B)\}$	by 1st De Morgan law for Prop Logic
$= \{x \mid x \notin A \vee x \notin B\}$	by defn. of not belong symbol
$= \{x \mid x \in \overline{A} \vee x \in \overline{B}\}$	by defn. of complement
$= \{x \mid x \in \overline{A} \cup \overline{B}\}$	by defn. of union
$= \overline{A} \cup \overline{B}$	by meaning of notation

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Membership Table

- **Example**: Construct a membership table to show that the distributive law holds.

$$A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$$

- **Solution**:

A	B	C	B ∩ C	A ∪ (B ∩ C)	A ∪ B	A ∪ C	(A ∪ B) ∩ (A ∪ C)
1	1	1	1	1	1	1	1
1	1	0	0	1	1	1	1
1	0	1	0	1	1	1	1
1	0	0	0	1	1	1	1
0	1	1	1	1	1	1	1
0	1	0	0	0	1	0	0
0	0	1	0	0	0	1	0
0	0	0	0	0	0	0	0

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Generalized Unions and Intersections

- Let $A_1, A_2, \ldots, A_n$ be an indexed collection of sets.
- We define: $\bigcup_{i=1}^{n} A_i = A_1 \cup A_2 \cup \ldots \cup A_n$

$$\bigcap_{i=1}^{n} A_i = A_1 \cap A_2 \cap \ldots \cap A_n$$

- These are well defined, since union and intersection are associative.
- For $i = 1,2,\ldots$, let $A_i = \{i, i + 1, i + 2, \ldots\}$. Then,

$$\bigcup_{i=1}^{n} A_i = \bigcup_{i=1}^{n} \{i, i+1, i+2, \ldots\} = \{1, 2, 3, \ldots\}$$

$$\bigcap_{i=1}^{n} A_i = \bigcap_{i=1}^{n} \{i, i+1, i+2, \ldots\} = \{n, n+1, n+2, \ldots\} = A_n$$

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Functions

- Section 2.3

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Section Summary

- Definition of a Function
  - Domain, Codomain
  - Image, Preimage
- Injection, Surjection, Bijection
- Inverse Function
- Function Composition
- Graphing Functions
- Floor, Ceiling, Factorial
- Partial Functions (optional)

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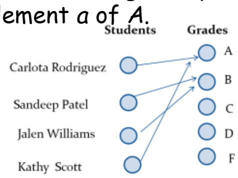
Functions

- **Definition**: Let *A* and *B* be nonempty sets. A *function f* from *A* to *B*, denoted *f*: $A \to B$ is an assignment of each element of *A* to exactly one element of *B*. We write *f*(*a*) = *b* if *b* is the unique element of *B* assigned by the function f to the element *a* of *A*.
  - Functions are sometimes called *mappings* or *transformations*.



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Functions

- A function f: $A \to B$ can also be defined as a subset of $A \times B$ (a relation). This subset is restricted to be a relation where no two elements of the relation have the same first element.
- Specifically, a function *f* from *A* to *B* contains one, and only one ordered pair (*a*, *b*) for every element *a*∈ *A*. $\forall x \left[x \in A \to \exists y \left[y \in B \wedge (x, y) \in f \right] \right]$
- and

$$\forall x, y_1, y_2 \left[\left[(x, y_1) \in f \wedge (x, y_2) \in f \right] \to y_1 = y_2 \right]$$

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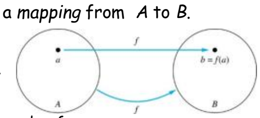
Functions

Given a function *f*: $A \to B$:

- We say *f maps A* to *B or f* is a *mapping* from *A* to *B*.
- *A* is called the *domain* of *f*.
- *B* is called the *codomain* of *f*.
- If *f*(*a*) = *b*,
  - then *b* is called the *image* of *a* under *f*.
  - *a* is called the *preimage* of *b*.
- The range of *f* is the set of all images of points in **A** under *f*. We denote it by *f*(**A**).
- Two functions are *equal* when they have the same domain, the same codomain and map each element of the domain to the same element of the codomain.



Jump to long description

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Representing Functions

- functions may be specified in different ways
  - an explicit statement of the assignment
    - example: students and grades
  - a formula
    - example: f(x) = x + 1
  - a computer program
    - example: a Java program that when given an integer n, produces the nth Fibonacci Number

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Practice Questions

f(a) = ? z

The image of d is ? z

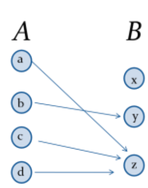
The domain of f is ? A

The codomain of f is ? B

The preimage of y is ? b

The range of *f*, *f*(*A*) = ? {y, z}

The preimage(s) of z is (are) ? {a, c, d}



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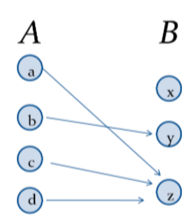
Questions on Functions and Sets

- if f: A →B and S is a subset of A, then

  f(S) = {f(s) | s ∈ S}

- f{a, b, c} is ? {y,z}
- f{c, d} is ? {z}



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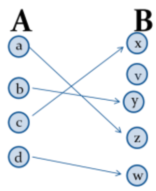
Injections (one-to-one)

Definition: A function f is said to be one-to-one, or injective, if and only if f(a) = f(b) implies that a = b for all a and b in the domain of f.

- i.e., each element in the codomain has no more than 1 arrow pointing to it



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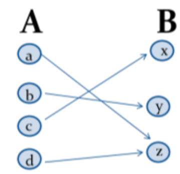
Surjections (onto)

Definition: A function f from A to B is called onto, or surjective, if and only if for every element $b \in B$ there is an element $a \in A$ with f(a) = b

- i.e., every element in B has at least one arrow pointing to it



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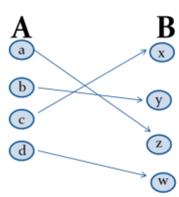
Bijections

- **Definition**: A function f is a bijection if it is both one-to-one and onto (surjective and injective).
  - also termed one-to-one correspondence



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Showing f Is one-to-one Or onto

- suppose that $f : A \to B$
- to show that f is injective
  - show that if f(x) = f(y) for arbitrary $x, y \in A$, then x = y
- to show that f is not injective
  - find particular elements $x, y \in A$ such that $x \neq y$ and f(x) = f(y)
- to show that f is surjective
  - consider an arbitrary element $y \in B$ and find an element $x \in A$ such that f(x) = y
- to show that f is not surjective
  - find a particular $y \in B$ such that $f(x) \neq y$ for all $x \in A$

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Showing f Is one-to-one Or onto

- **Example 1**: Let f be the function from {a, b, c, d} to {1, 2, 3} defined by f(a) = 3, f(b) = 2, f(c) = 1, and f(d) = 3. Is f an onto function?
- **Solution**: Yes, f is onto since all three elements of the codomain are images of elements in the domain. If the codomain were changed to {1, 2, 3, 4}, f would not be onto.
- **Example 2**: Is the function $f(x) = x^2$ from the set of integers to the set of integers onto?
- **Solution**: No, f is not onto because there is no integer x with $x^2 = -1$, for example.

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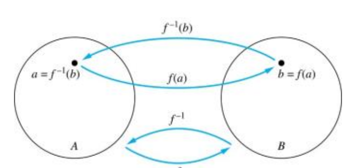
Inverse Functions

Definition: Let f be a bijection from A to B. Then the inverse of f, denoted f^{-1}, is the function from B to A defined as $f^{-1}(y) = x$ iff f(x) = y.

- no inverse exists unless f is a bijection (why?)



Jump to long description

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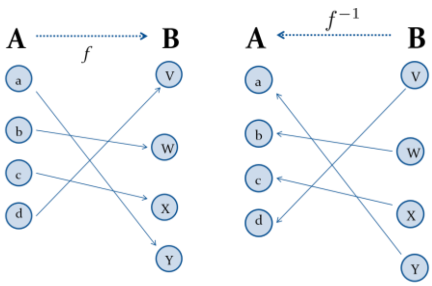
Inverse Functions



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Questions

- **Example 1**: Let f be the function from {a,b,c} to {1, 2, 3} such that f(a) = 2, f(b) = 3, and f(c) = 1. Is f invertible and if so, what is its inverse?

- **Solution**: The function f is invertible because it is a one-to-one correspondence. The inverse function f^{-1} reverses the correspondence given by f, so $f^{-1}(1) = c$, $f^{-1}(2) = a$, and $f^{-1}(3) = b$.

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Questions

- **Example 2**: Let f: **Z** → **Z** be such that f(x) = x + 1. Is f invertible, and if so, what is its inverse?

- **Solution**: The function f is invertible because it is a one-to-one correspondence. The inverse function f^{-1} reverses the correspondence so $f^{-1}(y) = y - 1$.

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Questions

Example 3: Let f: **R** → **R** be such that $f(x) = x^2$. Is f invertible, and if so, what is its inverse?

Solution: The function f is not invertible because it is not one-to-one.

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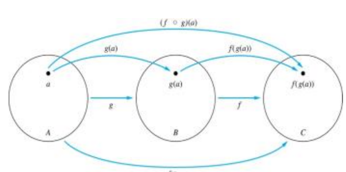
Composition

Definition: Let f: B→C and g: A→B. The composition of f with g, denoted f ∘ g, is the function from A to C defined by

$$(f \circ g)(x) = f(g(x))$$



Jump to long description

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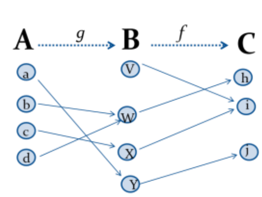
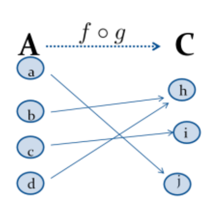
Composition





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Composition

- **Example 1**: If f(x) = x^2 and g(x) = 2x + 1, then

  f(g(x)) = $(2x + 1)^2$

  and

  g(f(x)) = $2x^2 + 1$

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Composition Questions

- **Example 2**: Let g be the function from the set {a, b, c} to itself such that g(a) = b, g(b) = c, and g(c) = a. Let f be the function from the set {a, b, c} to the set {1, 2, 3} such that f(a) = 3, f(b) = 2, and f(c) = 1.
- What is the composition of f and g, and what is the composition of g and f?
- **Solution**: The composition f ∘ g is defined by

$$f \circ g(a) = f(g(a)) = f(b) = 2.$$
$$f \circ g(b) = f(g(b)) = f(c) = 1.$$
$$f \circ g(c) = f(g(c)) = f(a) = 3.$$

Note that g ∘ f is not defined, because the range of f is not a subset of the domain of g.

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Composition Questions

- **Example 2**: Let f and g be functions from the set of integers to the set of integers defined by

$$f(x) = 2x + 3 \quad \text{and} \quad g(x) = 3x + 2.$$

- What is the composition of f and g, and also the composition of g and f ?
- **Solution:**

$$f \circ g\ (x) = f(g(x)) = f(3x+2) = 2(3x+2)+3 = 6x + 7$$

$$g \circ f\ (x) = g(f(x)) = g(2x+3) = 3(2x+3)+2 = 6x+11$$

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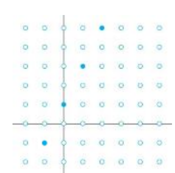
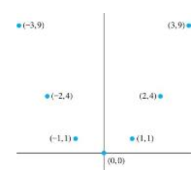
Graphs of Functions

Let f be a function from the set A to the set B. The graph of the function f is the set of ordered pairs $\{(a,b) \mid a \in A \text{ and } f(a) = b\}$.



Graph of f(n) = 2n + 1 from Z to Z



Graph of f(x) = x^2 from Z to Z

Jump to long description

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Some Important Functions

The floor function, denoted

$$f(x) = \lfloor x \rfloor$$

is the largest integer less than or equal to x

The ceiling function, denoted

$$f(x) = \lceil x \rceil$$

is the smallest integer greater than or equal to x

Example: $\lceil 3.5 \rceil = 4 \qquad \lfloor 3.5 \rfloor = 3$

$\lceil -1.5 \rceil = -1 \qquad \lfloor -1.5 \rfloor = -2$

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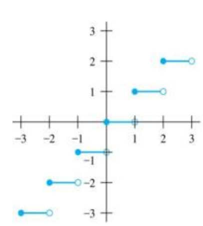
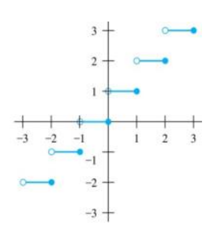
Floor and Ceiling Functions



(a) $y = \lfloor x \rfloor$



(b) $y = \lceil x \rceil$

- Graph of (a) Floor and (b) Ceiling Functions

Jump to long description

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Factorial Function

Definition: f: **N** → **Z**⁺, denoted by f(n) = n! is the product of the first n positive integers when n is a nonnegative integer.

$$f(n) = 1 \cdot 2 \cdots (n-1) \cdot n, \qquad f(0) = 0! = 1$$

Stirling's Formula:

$n! \sim \sqrt{2\pi n}(n/e)^n$

$f(n) \sim g(n) \doteq lim_{n\to\infty} f(n)/g(n) = 1$

Examples:

$f(1) = 1! = 1$

$f(2) = 2! = 1 \cdot 2 = 2$

$f(6) = 6! = 1 \cdot 2 \cdot 3 \cdot 4 \cdot 5 \cdot 6 = 720$

$f(20) = 2{,}432{,}902{,}008{,}176{,}640{,}000.$

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Sequences and Summations

- Section 2.4

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Section Summary

- Sequences.
  - Examples: Geometric Progression, Arithmetic Progression
- Recurrence Relations
  - Example: Fibonacci Sequence
- Summations
- Special Integer Sequences (*optional*)

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Introduction

- Sequences are ordered lists of elements.
  - 1, 2, 3, 5, 8
  - 1, 3, 9, 27, 81, …….
- Sequences arise throughout mathematics, computer science, and in many other disciplines, ranging from botany to music.
- We will introduce the terminology to represent sequences and sums of the terms in the sequences.

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Sequences

- **Definition**: A sequence is a function from a subset of the integers (usually either the set {0, 1, 2, 3, 4, …..} or {1, 2, 3, 4, ….}) to a set S.
- The notation a_n is used to denote the image of the integer n. We can think of a_n as the equivalent of f(n) where f is a function from {0,1,2,…..} to S. We call a_n a term of the sequence.

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Sequences

- **Example**: Consider the sequence $\{a_n\}$ where

$$a_n = \frac{1}{n} \qquad \{a_n\} = \{a_1, a_2, a_3 \ldots\}$$

$$1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}$$

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Geometric Progression

- **Definition**: A geometric progression is a sequence of the form: $a, ar^1, ar^2, \ldots, ar^n, \ldots$
- where the initial term a and the common ratio r are real numbers.

Examples :

1. Let $a = 1$ and $r = -1$. Then :
$$\{b_n\} = \{b_0, b_1, b_2, b_3, b_4, \ldots\} = \{1, -1, 1, -1, 1, \ldots\}$$
2. Let $a = 2$ and $r = 5$. Then :
$$\{c_n\} = \{c_0, c_1, c_2, c_3, c_4, \ldots\} = \{2, 10, 50, 250, 1250, \ldots\}$$
3. Let $a = 6$ and $r = 1/3$. Then :
$$\{d_n\} = \{d_0, d_1, d_2, d_3, d_4, \ldots\} = \left\{6, 2, \frac{2}{3}, \frac{2}{9}, \frac{2}{27}, \ldots\right\}$$

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Arithmetic Progression

- **Definition:** A arithmetic progression is a sequence of the form: $a, a+d, a+2d, \ldots, a+nd, \ldots$
- where the initial term a and the common difference d are real numbers.

Examples :

1. Let $a = -1$ and $d = 4$:
$$\{s_n\} = \{s_0, s_1, s_2, s_3, s_4, \ldots\} = \{-1, 3, 7, 11, 15, \ldots\}$$
2. Let $a = 7$ and $d = -3$:
$$\{t_n\} = \{t_0, t_1, t_2, t_3, t_4, \ldots\} = \{7, 4, 1, -2, -5, \ldots\}$$
3. Let $a = 1$ and $d = 2$:
$$\{u_n\} = \{u_0, u_1, u_2, u_3, u_4, \ldots\} = \{1, 3, 5, 7, 9, \ldots\}$$

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Strings

- **Definition**: A string is a finite sequence of characters from a finite set (an alphabet).
- Sequences of characters or bits are important in computer science.
- The empty string is represented by λ.
- The string abcde has length 5.

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Recurrence Relations

- **Definition:** A recurrence relation for the sequence $\{a_n\}$ is an equation that expresses a_n in terms of one or more of the previous terms of the sequence, namely, $a_0, a_1, \ldots, a_{n-1}$, for all integers n with $n \geq n_0$, where n_0 is a nonnegative integer.
- A sequence is called a solution of a recurrence relation if its terms satisfy the recurrence relation.
- The initial conditions for a sequence specify the terms that precede the first term where the recurrence relation takes effect.

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Questions about Recurrence Relations

- **Example** 1: Let $\{a_n\}$ be a sequence that satisfies the recurrence relation $a_n = a_{n-1} + 3$ for n = 1,2,3,4,…. and suppose that $a_0 = 2$. What are a_1 , a_2 and a_3?

  [Here $a_0 = 2$ is the initial condition.]
- **Solution**: We see from the recurrence relation that

$$a_1 = a_0 + 3 = 2 + 3 = 5$$
$$a_2 = 5 + 3 = 8$$
$$a_3 = 8 + 3 = 11$$

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Questions about Recurrence Relations

- **Example** 2: Let $\{a_n\}$ be a sequence that satisfies the recurrence relation $a_n = a_{n-1} - a_{n-2}$ for n = 2,3,4,…. and suppose that $a_0 = 3$ and $a_1 = 5$. What are a_2 and a_3?
- [Here the initial conditions are $a_0 = 3$ and $a_1 = 5$.]
- **Solution**: We see from the recurrence relation that

$$a_2 = a_1 - a_0 = 5 - 3 = 2$$
$$a_3 = a_2 - a_1 = 2 - 5 = -3$$

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Fibonacci Sequence

- **Definition:** Define the Fibonacci sequence, $f_0, f_1, f_2, \ldots$, by
  - initial conditions: $f_0 = 0$, $f_1 = 1$
  - recurrence relation: $f_n = f_{n-1} + f_{n-2}$
- **Example**: Find f_2, f_3, f_4, f_5 and f_6.

Answer :

$$f_2 = f_1 + f_0 = 1 + 0 = 1,$$
$$f_3 = f_2 + f_1 = 1 + 1 = 2,$$
$$f_4 = f_3 + f_2 = 2 + 1 = 3,$$
$$f_5 = f_4 + f_3 = 3 + 2 = 5,$$
$$f_6 = f_5 + f_4 = 5 + 3 = 8.$$

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Solving Recurrence Relations

- Finding a formula for the n^{th} term of the sequence generated by a recurrence relation is called solving the recurrence relation.
- Such a formula is called a closed, or closed-form, formula.
- Various methods for solving recurrence relations will be covered in Chapter 8.
- Here we illustrate by example the method of iteration in which we need to guess the formula. The guess can be proved correct by induction (Chapter 5).

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Iterative Solution Example

- **Method 1**: Working upward, forward substitution. Let $\{a_n\}$ be a sequence that satisfies the recurrence relation $a_n = a_{n-1} + 3$ for $n = 2,3,4,\ldots$ and suppose that $a_1 = 2$.

$a_2 = 2 + 3$

$a_3 = (2 + 3) + 3 = 2 + 3 \cdot 2$

$a_4 = (2 + 3 \cdot 2) + 3 = 2 + 3 \cdot 3$

.

.

.

$a_n = a_{n-1} + 3 = (2 + 3 \cdot (n-2)) + 3 = 2 + 3(n-1)$

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Iterative Solution Example

- **Method 2**: Working downward, backward substitution. Let $\{a_n\}$ be a sequence that satisfies the recurrence relation $a_n = a_{n-1} + 3$ for $n = 2,3,4,\ldots$ and suppose that $a_1 = 2$.
  - $a_n = a_{n-1} + 3$
    - $= (a_{n-2} + 3) + 3 = a_{n-2} + 3 \cdot 2$

      $= (a_{n-3} + 3) + 3 \cdot 2 = a_{n-3} + 3 \cdot 3$

      .

      .

      .
    - $= a_2 + 3(n-2) = (a_1 + 3) + 3(n-2) = 2 + 3(n-1)$

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Financial Application

- **Example**: Suppose that a person deposits \$10,000.00 in a savings account at a bank yielding 11% per year with interest compounded annually. How much will be in the account after n years?
- Let P_n denote the amount in the account after n years. P_n satisfies the following recurrence relation:

$$P_n = P_{n-1} + 0.11P_{n-1} = (1.11)\, P_{n-1}$$

with the initial condition $P_0 = 10{,}000$

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Financial Application

- $P_n = P_{n-1} + 0.11P_{n-1} = (1.11)\, P_{n-1}$

  with the initial condition $P_0 = 10{,}000$
- **Solution**: Forward Substitution

  $P_1 = (1.11)P_0$

  $P_2 = (1.11)P_1 = (1.11)^2P_0$

  $P_3 = (1.11)P_2 = (1.11)^3P_0$

  :

  $P_n = (1.11)P_{n-1} = (1.11)^nP_0 = (1.11)^n\, 10{,}000$

  $P_n = (1.11)^n\, 10{,}000$

  $P_{30} = (1.11)^{30}\, 10{,}000 = \$228{,}992.97$

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Special Integer Sequences

- Given a few terms of a sequence, try to identify the sequence. Conjecture a formula, recurrence relation, or some other rule.
- some questions to ask
  - are there repeated terms of the same value?
  - can you obtain a term from the previous term by adding an amount or multiplying by an amount?
  - can you obtain a term by combining the previous terms in some way?
  - are there cycles among the terms?
  - do the terms match those of a well known sequence?

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Questions on Special Integer Sequences

- **Example 1**: Find formula for the sequences with the following first five terms: 1, ½, ¼, 1/8, 1/16

  Solution: Note that the denominators are powers of 2. The sequence with $a_n = 1/2^n$ is a possible match. This is a geometric progression with a = 1 and r = ½.
- **Example 2**: Consider 1,3,5,7,9

  Solution: Note that each term is obtained by adding 2 to the previous term. A possible formula is $a_n = 2n + 1$. This is an arithmetic progression with a =1 and d = 2.
- **Example 3**: 1, -1, 1, -1,1

  Solution: The terms alternate between 1 and -1. A possible sequence is $a_n = (-1)^n$. This is a geometric progression with a = 1 and r = −1.

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Questions on Special Integer Sequences

- **TABLE 1** Some Useful Sequences.

nth Term	*First 10 Terms*
n^2	1, 4, 9, 16, 25, 36, 49, 64, 81, 100,…
n^3	1, 8, 27, 64, 125, 216, 343, 512, 729, 1000,…
n^4	1, 16, 81, 256, 625, 1296, 2401, 4096, 6561, 10000,…
f_n	1, 1, 2, 3, 5, 8, 13, 21, 34, 55, 89,…
2^n	2, 4, 8, 16, 32, 64, 128, 256, 512, 1024,…
3^n	3, 9, 27, 81, 243, 729, 2187, 6561, 19683, 59049,…
$n!$	1, 2, 6, 24, 120, 720, 5040, 40320, 362880, 3628800,…

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Guessing Sequences

- **Example**: Conjecture a simple formula for a_n if the first 10 terms of the sequence $\{a_n\}$ are 1, 7, 25, 79, 241, 727, 2185, 6559, 19681, 59047.

  Solution: Note the ratio of each term to the previous approximates 3. So now compare with the sequence 3^n . We notice that the n^{th} term is 2 less than the corresponding power of 3. So a good conjecture is that $a_n = 3^n - 2$.

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Integer Sequences

- integer sequences appear in a wide range of contexts
  - sequence of prime numbers (Chapter 4)
  - number of ways to order n discrete objects (Chapter 6)
  - number of moves needed to solve the Tower of Hanoi puzzle with n disks (Chapter 8)
  - number of rabbits on an island over time (Chapter 8)
- integer sequences are useful in many fields such as biology, engineering, chemistry and physics
- On-Line Encyclopedia of Integer Sequences (OESIS) contains over 200,000 sequences http://oeis.org/Spuzzle.html

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Integer Sequences

- Here are three interesting sequences to try from the OESIS site. To solve each puzzle, find a rule that determines the terms of the sequence.
- Guess the rules for forming for the following sequences:
  - 2, 3, 3, 5, 10, 13, 39, 43, 172, 177, ...
    - Hint: Think of adding and multiplying by numbers to generate this sequence.
  - 0, 0, 0, 0, 4, 9, 5, 1, 1, 0, 55, ...
    - Hint: Think of the English names for the numbers representing the position in the sequence and the Roman Numerals for the same number.
  - 2, 4, 6, 30, 32, 34, 36, 40, 42, 44, 46, ...
    - Hint: Think of the English names for numbers, and whether or not they have the letter 'e'
- The answers and many more can be found at http://oeis.org/Spuzzle.html

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Summations

Sum of the terms $a_m, a_m + 1, \ldots, a_n$

from the sequence $\{a_n\}$

The notation:

$$\sum_{j=m}^{n} a_j \qquad \sum_{j=m}^{n} a_j \qquad \sum_{m \le j \le n} a_j$$

represents

$$a_m + a_{m+1} + \cdots + a_n$$

- The variable j is called the index of summation. It runs through all the integers starting with its lower limit m and ending with its upper limit n.

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Summations

More generally for a set S:

$$\sum_{j \in s} a_j$$

Examples:

$$r^0 + r^1 + r^2 + r^3 + \cdots + r^n = \sum_{0}^{n} r^j$$

$$1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \cdots = \sum_{1}^{\infty} \frac{1}{i}$$

If $S = \{2, 5, 7, 10\}$ then $\sum_{j \in s} a_j = a_2 + a_5 + a_7 + a_{10}$

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Summation Properties

- scalar product: can move scalar outside summation

$$\sum_{i=1}^{3} 2i = 2\sum_{i=1}^{3} i$$

- addition: can separate

$$\sum_{i=1}^{3} (i+2) = \sum_{i=1}^{3} i + \sum_{i=1}^{3} 2$$

- sum of one step

$$\sum_{i=3}^{3} i = 3$$

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Summation Properties

- sum of scalars: multiply by number of steps (n – m + 1)

$$\sum_{i=2}^{4} 2 = 6$$

- example

$$\sum_{i=2}^{4} (3i+4) =$$

$$3\sum_{i=2}^{4} i + \sum_{i=2}^{4} 4 =$$

3(2 + 3 + 4) + 4(4 – 2 + 1) = 3(9) + 4(3) = 39

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Summation Properties

- nested: work from innermost to outermost

$$\sum_{i=1}^{3}\sum_{j=2}^{4} (i+j) =$$

$$\sum_{i=1}^{3} (i+2+i+3+i+4) =$$

$$\sum_{i=1}^{3} (3i+9) =$$

$$3\sum_{i=1}^{3} i + \sum_{i=1}^{3} 9 = 3(1+2+3) + 9(3) = 45$$

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Product Notation (optional)

Product of the terms $a_m, a_m + 1, \ldots, a_n$

from the sequence $\{a_n\}$

The notation:

$$\prod_{j=m}^{n} a_j \qquad \prod_{j=m}^{n} a_j \qquad \prod_{m \le j \le n} a_j$$

represents

$$a_m \times a_{m+1} \times \cdots \times a_n$$

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Geometric Series

Sums of terms of geometric progressions

$$\sum_{j=0}^{n} ar^j = \begin{cases} \dfrac{ar^{n+1}-a}{r-1} & r \neq 1 \\ (n+1)a & r = 1 \end{cases}$$

Proof: Let $S_n = \sum_{j=0}^{n} ar^j$

$rS_n = r\sum_{j=0}^{n} ar^j$

$= \sum_{j=0}^{n} ar^{j+1}$

To compute S_n, first multiply both sides of the equality by r and then manipulate the resulting sum as follows:

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Geometric Series

$= \sum_{j=0}^{n} ar^{j+1}$ From previous slide.

$= \sum_{k=1}^{n+1} ar^k$ Shifting the index of summation with $k = j + 1$.

$= \left(\sum_{k=0}^{n} ar^k\right) + \left(ar^{n+1} - a\right)$ Removing $k = n + 1$ term and adding $k = 0$ term.

$= S_n + \left(ar^{n+1} - a\right)$ Substituting S for summation formula

$\therefore \quad rS_n = S_n + \left(ar^{n+1} - a\right)$

$S_n = \dfrac{ar^{n+1} - a}{r-1}$ if $r \neq 1$

$S_n = \sum_{j=0}^{n} ar^j = \sum_{j=0}^{n} a = (n+1)a$ if r = 1

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Some Useful Summation Formulae

TABLE 2 Some Useful Summation Formulae.

Sum	*Closed From*
$\sum_{k=0}^{n} ar^k \ (r \neq 0)$	$\dfrac{ar^{n+1}-a}{r-1}, r \neq 1$
$\sum_{k=1}^{n} k$	$\dfrac{n(n+1)}{2}$
$\sum_{k=1}^{n} k^2$	$\dfrac{n(n+1)(2n+1)}{6}$
$\sum_{k=1}^{n} k^3$	$\dfrac{n^2(n+1)^2}{4}$
$\sum_{k=0}^{\infty} x^k, \lvert x \rvert < 1$	$\dfrac{1}{1-x}$
$\sum_{k=0}^{\infty} kx^{k-1}, \lvert x \rvert < 1$	$\dfrac{1}{(1-x)^2}$

Geometric Series: we just proved this

Later we will prove some of these by induction.

Proof in text (requires calculus)

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Cardinality of Sets

- Section 2.5

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Section Summary

- Cardinality
- Countable Sets
- Computability

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Cardinality

- **Definition**: The cardinality of a set A is equal to the cardinality of a set B, denoted |A| = |B|,
  - if and only if there is a one-to-one correspondence (i.e., a bijection) from A to B
  - if there is a one-to-one function (i.e., an injection) from A to B, the cardinality of A is less than or the same as the cardinality of B and we write |A| ≤ |B|
  - when |A| ≤ |B| and A and B have different cardinality, we say that the cardinality of A is less than the cardinality of B and write |A| < |B|

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Cardinality

- **Definition**: A set that is either finite or has the same cardinality as the set of positive integers (**Z**$^+$) is called countable. A set that is not countable is uncountable.
- The set of real numbers **R** is an uncountable set.
- When an infinite set is countable (countably infinite), its cardinality is $\aleph_0$ (where $\aleph$ is aleph, the 1st letter of the Hebrew alphabet). We write $|S| = \aleph_0$ and say that S has cardinality "aleph null."

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Showing that a Set is Countable

- An infinite set is countable if and only if it is possible to list the elements of the set in a sequence (indexed by the positive integers).
- In other words, a one-to-one correspondence f from the set of positive integers to a set S can be expressed in terms of a sequence $a_1, a_2, \ldots, a_n, \ldots$ where $a_1 = f(1)$, $a_2 = f(2), \ldots, a_n = f(n), \ldots$

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Hilbert's Grand Hotel

David Hilbert

The Grand Hotel (example due to David Hilbert) has a countably infinite number of rooms, each occupied by a guest. We can always accommodate a new guest at this hotel. How is this possible?

Explanation: Because the rooms of Grand Hotel are countable, we can list them as Room 1, Room 2, Room 3, and so on. When a new guest arrives, we move the guest in Room 1 to Room 2, the guest in Room 2 to Room 3, and in general the guest in Room n to Room n + 1, for all positive integers n. This frees up Room 1, which we assign to the new guest, and all the current guests still have rooms.

The hotel can also accommodate a countable number of new guests, and all the guests on a countable number of buses where each bus contains a countable number of guests.

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Showing that a Set is Countable

- **Example 1**: Show that the set of positive even integers E is countable set.

  Solution: Let $f(x) = 2x$.

$$\begin{array}{cccccc} 1 & 2 & 3 & 4 & 5 & 6 \\ \updownarrow & \updownarrow & \updownarrow & \updownarrow & \updownarrow & \updownarrow \\ 2 & 4 & 6 & 8 & 10 & 12 \end{array}$$

  Then f is a bijection from **N** to E since f is both one-to-one and onto. To show that it is one-to-one, suppose that $f(n) = f(m)$. Then $2n = 2m$, and so $n = m$. To see that it is onto, suppose that t is an even positive integer. Then $t = 2k$ for some positive integer k and $f(k) = t$.

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Showing that a Set is Countable

- **Example 2**: Show that the set of integers **Z** is countable.

  Solution: Can list in a sequence:

  0, 1, − 1, 2, − 2, 3, − 3 ,..........

  Or can define a bijection from **N** to **Z**:

  When n is even: $f(n) = n/2$

  When n is odd: $f(n) = -(n-1)/2$

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The Positive Rational Numbers are Countable

- **Definition**: A rational number can be expressed as the ratio of two integers p and q such that $q \neq 0$.
  - $\frac{3}{4}$ is a rational number
  - $\sqrt{2}$ is not a rational number.
- **Example 3**: Show that the positive rational numbers are countable.

  Solution: The positive rational numbers are countable since they can be arranged in a sequence:

  $r_1, r_2, r_3, \ldots$

  The next slide shows how this is done.

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The Positive Rational Numbers are Countable

Constructing the List

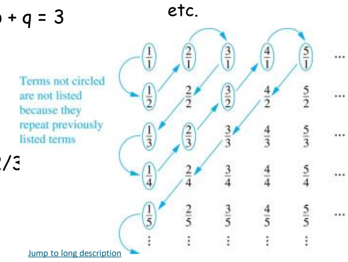
First list p/q with p + q = 2.

Next list p/q with p + q = 3

And so on.

1, ½, 2, 3, 1/3,1/4, 2/3

First row q = 1.
Second row q = 2.
etc.



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Strings

- **Example 4**: Show that the set of finite strings S over a finite alphabet A is countably infinite.

  Assume an alphabetical ordering of symbols in A

  Solution: Show that the strings can be listed in a sequence. First list
  1. All the strings of length 0 in alphabetical order.
  2. Then all the strings of length 1 in lexicographic (as in a dictionary) order.
  3. Then all the strings of length 2 in lexicographic order.
  4. And so on.

  This implies a bijection from **N** to S and hence it is a countably infinite set.

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The Set of all Java Programs is Countable

- **Example 5**: Show that the set of all Java programs is countable.
- **Solution**: Let *S* be the set of strings constructed from the characters which can appear in a Java program. Use the ordering from the previous example. Take each string in turn:
  - Feed the string into a Java compiler. (A Java compiler will determine if the input program is a syntactically correct Java program.)
  - If the compiler says YES, this is a syntactically correct Java program, we add the program to the list.
  - We move on to the next string.
- In this way we construct an implied bijection from **N** to the set of Java programs. Hence, the set of Java programs is countable.

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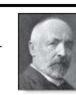
The Real Numbers are Uncountable

Georg Cantor (1845-1918)



Example: Show that the set of real numbers is uncountable.

Solution: The method is called the Cantor diagonalization argument, and is a proof by contradiction.

1. Suppose **R** is countable. Then the real numbers between 0 and 1 are also countable (any subset of a countable set is countable - an exercise in the text).
2. The real numbers between 0 and 1 can be listed in order $r_1, r_2, r_3, \ldots$.
3. Let the decimal representation of this listing be
   $$r_1 = 0.d_{11}d_{12}d_{13}d_{14}d_{15}d_{16}\ldots$$
   $$r_2 = 0.d_{21}d_{22}d_{23}d_{24}d_{25}d_{26}\ldots$$
   $$\vdots$$
4. Form a new real number with the decimal expansion $r = .r_1r_2r_3r_4\ldots$
   where $r_i = 3 \text{ if } d_{ii} \neq 3$ and $r_i = 4 \text{ if } d_{ii} = 3$
5. *r* is not equal to any of the $r_1, r_2, r_3, \ldots$ Because it differs from r_i in its *i*th position after the decimal point. Therefore there is a real number between 0 and 1 that is not on the list since every real number has a unique decimal expansion. Hence, all the real numbers between 0 and 1 cannot be listed, so the set of real numbers between 0 and 1 is uncountable.
6. Since a set with an uncountable subset is uncountable (an exercise), the set of real numbers is uncountable.

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Matrices

- Section 2.6

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Section Summary

- Definition of a Matrix
- Matrix Arithmetic
- Transposes and Powers of Arithmetic
- Zero-One matrices

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Matrices

- matrices are useful discrete structures that can be used in many ways, e.g., they are used to
  - describe certain types of functions known as linear transformations
  - example: express which vertices of a graph are connected by edges (see Chapter 10)
- in later chapters, we will see matrices used to build models of
  - transportation systems
  - communication networks
- here we cover the aspect of matrix arithmetic that will be needed later

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Matrix

- **Definition**: A matrix is a rectangular array of numbers.
  - a matrix with m rows and n columns is called an $m \times n$ matrix
  - the plural of matrix is matrices
  - a matrix with the same number of rows as columns is called square
  - two matrices are equal if they have the same number of rows and the same number of columns and the corresponding entries in every position are equal.
    - 3×2 matrix $\begin{bmatrix} 1 & 1 \\ 0 & 2 \\ 1 & 3 \end{bmatrix}$

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Notation

Let m and n be positive integers and let $A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ . & . & . & . \\ . & . & . & . \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix}$

The i^{th} row of **A** is the $1 \times n$ matrix $[a_{i1}, a_{i2},\ldots,a_{in}]$. The j^{th} column of **A** is the $m \times 1$ matrix: $\begin{bmatrix} a_{1j} \\ a_{2j} \\ . \\ . \\ a_{mj} \end{bmatrix}$

The (i,j)th element or entry of **A** is the element a_{ij}. We can use **A** = $[a_{ij}]$ to denote the matrix with its (i,j)th element equal to a_{ij}.

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Matrix Arithmetic: Addition

- **Definition**: Let **A** = $[a_{ij}]$ and **B** = $[b_{ij}]$ be $m \times n$ matrices. The sum of **A** and **B**, denoted by **A + B**, is the $m \times n$ matrix that has $a_{ij} + b_{ij}$ as its (i,j)th element. In other words, **A + B** = $[a_{ij} + b_{ij}]$.
- **Example**:

$$\begin{bmatrix} 1 & 0 & -1 \\ 2 & 2 & -3 \\ 3 & 4 & 0 \end{bmatrix} + \begin{bmatrix} 3 & 4 & -1 \\ 1 & -3 & 0 \\ -1 & 1 & 2 \end{bmatrix} = \begin{bmatrix} 4 & 4 & -2 \\ 3 & -1 & -3 \\ 2 & 5 & 2 \end{bmatrix}$$

Note that matrices of different sizes cannot be added.

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Matrix Multiplication

- **Definition**: Let **A** be an $m \times k$ matrix and **B** be a $k \times n$ matrix. The product of **A** and **B**, denoted by **AB**, is the $m \times n$ matrix that has its (i,j)th element equal to the sum of the products of the corresponding elements from the i^{th} row of **A** and the j^{th} column of **B**. In other words, if **AB** = $[c_{ij}]$ then $c_{ij} = a_{i1}b_{1j} + a_{i2}b_{2j} + \ldots + a_{kj}b_{2j}$.
- **Example**: $\begin{bmatrix} 1 & 0 & 4 \\ 2 & 1 & 1 \\ 3 & 1 & 0 \\ 0 & 2 & 2 \end{bmatrix} \begin{bmatrix} 2 & 4 \\ 1 & 1 \\ 3 & 0 \end{bmatrix} = \begin{bmatrix} 14 & 4 \\ 8 & 9 \\ 7 & 13 \\ 8 & 2 \end{bmatrix}$

The product of two matrices is undefined when the number of columns in the first matrix is not the same as the number of rows in the second.

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Illustration of Matrix Multiplication

- product of **A** = $[a_{ij}]$ and **B** = $[b_{ij}]$

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1k} \\ a_{21} & a_{22} & \cdots & a_{2k} \\ . & . & & . \\ . & . & & . \\ a_{i1} & a_{i2} & \cdots & a_{ik} \\ . & . & & . \\ . & . & & . \\ a_{m1} & a_{m2} & \cdots & a_{mk} \end{bmatrix} \qquad B = \begin{bmatrix} b_{11} & a_{12} & \cdots & b_{1j} & \cdots & b_{1n} \\ b_{21} & b_{22} & \cdots & b_{2j} & \cdots & b_{2n} \\ . & . & & . & & \\ . & . & & . & & \\ b_{k1} & b_{k2} & \cdots & b_{kj} & \cdots & b_{kn} \end{bmatrix}$$

$$AB = \begin{bmatrix} c_{11} & c_{12} & \cdots & c_{1n} \\ c_{21} & b_{22} & \cdots & c_{2n} \\ . & . & & . \\ . & . & c_{ij} & . \\ . & . & & . \\ c_{m1} & c_{m2} & \cdots & c_{mn} \end{bmatrix}$$

$$c_{ij} = a_{i1}b_{1j} + a_{i2}b_{2j} + \ldots + a_{ik}b_{kj}$$

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Matrix Multiplication is not Commutative

Example: Let

$$A = \begin{bmatrix} 1 & 1 \\ 2 & 1 \end{bmatrix} \qquad B = \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix}$$

Does **AB** = **BA**?

Solution:

$$AB = \begin{bmatrix} 3 & 2 \\ 5 & 3 \end{bmatrix} \qquad BA = \begin{bmatrix} 4 & 3 \\ 3 & 2 \end{bmatrix}$$

AB ≠ BA

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Identity Matrix and Powers of Matrices

Definition: The identity matrix of order n is the m × n matrix $\mathbf{I}_n = [\delta_{ij}]$, where $\delta_{ij} = 1$ if i = j and $\delta_{ij} = 0$ if i≠j.

$$I_n = \begin{bmatrix} 1 & 0 & \dots & 0 \\ 0 & 1 & \dots & 0 \\ . & . & & . \\ . & . & . & . \\ . & . & & . \\ 0 & 0 & \dots & 1 \end{bmatrix}$$

$\mathbf{AI}_n = \mathbf{I}_m\mathbf{A} = \mathbf{A}$

when **A** is an m × n matrix

Powers of square matrices can be defined. When A is an n × n matrix, we have:

$$A^0 = I_n \qquad A^r = \underbrace{AAA\cdots A}_{\text{r times}}$$

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Transposes of Matrices

- **Definition**: Let **A** = $[a_{ij}]$ be an m × n matrix. The transpose of **A**, denoted by $\mathbf{A}^T$, is the n × m matrix obtained by interchanging the rows and columns of **A**.

If $\mathbf{A}^T = [b_{ij}]$, then $b_{ij} = a_{ji}$ for i =1,2,…,n and j = 1,2, …,m.

$$\text{The transpose of the matrix} \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix} \text{is the matrix} \begin{bmatrix} 1 & 4 \\ 2 & 5 \\ 3 & 6 \end{bmatrix}$$

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Transposes of Matrices

- **Definition**: A square matrix **A** is called symmetric if $\mathbf{A} = \mathbf{A}^T$. Thus **A** = $[a_{ij}]$ is symmetric if $a_{ij} = a_{ji}$ for i and j with $1 \le i \le n$ and $1 \le j \le n$.

$$\text{The matrix} \begin{bmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} \text{is square}$$ (and symmetric)

- Symmetric matrices do not change when their rows and columns are interchanged.

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Zero-One Matrices

- **Definition**: A matrix all of whose entries are either 0 or 1 is called a zero-one matrix. (These will be used in Chapters 9 and 10.)
- Algorithms operating on discrete structures represented by zero-one matrices are based on Boolean arithmetic defined by the following Boolean operations:

$$b_1 \wedge b_2 = \begin{cases} 1 & \text{if } b_1 = b_2 = 1 \\ 0 & \text{otherwise} \end{cases} \qquad b_1 \vee b_2 = \begin{cases} 1 & \text{if } b_1 = 1 \text{ or } b_2 = 1 \\ 0 & \text{otherwise} \end{cases}$$

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Zero-One Matrices

- **Definition**: Let **A** = $[a_{ij}]$ and **B** = $[b_{ij}]$ be an m × n zero-one matrices.
  - The join of **A** and **B** is the zero-one matrix with (i,j)th entry $a_{ij} \vee b_{ij}$. The join of **A** and **B** is denoted by **A** ∨ **B**.
  - The meet of **A** and **B** is the zero-one matrix with (i,j)th entry $a_{ij} \wedge b_{ij}$. The meet of **A** and **B** is denoted by **A** ∧ **B**.

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Joins and Meets of Zero-One Matrices

Example: Find the join and meet of the zero-one matrices

$$A = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}, \qquad B = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 1 & 0 \end{bmatrix}.$$

Solution: The join of **A** and **B** is

$$A \vee B = \begin{bmatrix} 1 \vee 0 & 0 \vee 1 & 1 \vee 0 \\ 0 \vee 1 & 1 \vee 1 & 0 \vee 0 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 0 \end{bmatrix}$$

The meet of **A** and **B** is

$$A \wedge B = \begin{bmatrix} 1 \wedge 0 & 0 \wedge 1 & 1 \wedge 0 \\ 0 \wedge 1 & 1 \wedge 1 & 0 \wedge 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}.$$

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Boolean Product of Zero-One Matrices

- **Definition**: Let **A** = $[a_{ij}]$ be an m × k zero-one matrix and **B** = $[b_{ij}]$ be a k × n zero-one matrix. The Boolean product of **A** and **B**, denoted by **A** ⊙ **B**, is the m × n zero-one matrix with(i,j)th entry

$$c_{ij} = (a_{i1} \wedge b_{1j}) \vee (a_{i2} \wedge b_{2j}) \vee \ldots \vee (a_{ik} \wedge b_{kj}).$$

- **Example**: Find the Boolean product of **A** and **B**, where

$$A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 0 \end{bmatrix}, \qquad B = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix}.$$

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Boolean Product of Zero-One Matrices

- **Solution**: The Boolean product **A** ⊙ **B** is given by

$$A \odot B = \begin{bmatrix} (1 \wedge 1) \vee (0 \wedge 0) & (1 \wedge 1) \vee (0 \wedge 1) & (1 \wedge 0) \vee (0 \wedge 1) \\ (0 \wedge 1) \vee (1 \wedge 0) & (0 \wedge 1) \vee (1 \wedge 1) & (0 \wedge 0) \vee (1 \wedge 1) \\ (1 \wedge 1) \vee (0 \wedge 0) & (1 \wedge 1) \vee (0 \wedge 1) & (1 \wedge 0) \vee (0 \wedge 1) \end{bmatrix}$$

$$= \begin{bmatrix} 1 \vee 0 & 1 \vee 0 & 0 \vee 0 \\ 0 \vee 0 & 0 \vee 1 & 0 \vee 1 \\ 1 \vee 0 & 1 \vee 0 & 0 \vee 0 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 1 & 1 & 0 \end{bmatrix}.$$

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Boolean Powers of Zero-One Matrices

- **Definition**: Let **A** be a square zero-one matrix and let r be a positive integer. The r^th^ Boolean power of **A** is the Boolean product of r factors of **A**, denoted by $\mathbf{A}^{[r]}$. Hence,

$$A^{[r]} = \underbrace{A \odot A \odot \ldots \odot A}_{\text{r times}}.$$

- We define $\mathbf{A}^{[0]}$ to be $\mathbf{I}_n$.
- The Boolean product is well defined because the Boolean product of matrices is associative.

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Boolean Powers of Zero-One Matrices

- **Example**: Let

$$A = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 1 & 1 & 0 \end{bmatrix}.$$

- Find $\mathbf{A}^n$ for all positive integers n.
- **Solution**:

$$A^{[2]} = A \odot A = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 1 \end{bmatrix} \qquad A^{[3]} = A^{[2]} \odot A = \begin{bmatrix} 1 & 0 & 1 \\ 1 & 1 & 0 \\ 1 & 1 & 1 \end{bmatrix}$$

$$A^{[4]} = A^{[3]} \odot A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 1 \end{bmatrix}$$

$$A^{[5]} = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix} \qquad A^{[n]} = A^5 \text{ for all positive integers } n \text{ with } n \geq 5$$

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Appendix of Image Long Descriptions

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Functions₃ – Appendix

- The circle representing set A has element A inside. The circle representing set B has element B equals F left parenthesis A right parenthesis. Also, there are two arrows labeled F. From circle A to circle B, and from element A to B.

Jump to the image

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Inverse Functions₁ – Appendix

- There are two circles representing sets A and B. Circle A has element A equal to F power minus one left parenthesis B right parenthesis. Circle B has element B equal to F left parenthesis A right parenthesis. Also, there are 4 arrows: an arrow from element A to element B labeled F left parenthesis A right parenthesis, an arrow from element B to element A labeled F power minus one left parenthesis B right parenthesis, an arrow from circle A to circle B labeled F. And arrow from circle B to circle A labeled F power minus one.

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Composition₁ – Appendix

- There are three circles, representing sets A, B, and C. Circle A has element A. Circle B has element G left parenthesis A right parenthesis. Circle C has element F left parenthesis G left parenthesis A two right parentheses. Also, there are 6 arrows. From circle A to circle B labeled G. From circle B to circle C labeled F. From circle A to circle C labeled F circle G. From element A to element G left parenthesis A right parenthesis labeled G left parenthesis A right parenthesis. From element G left parenthesis A right parenthesis to element F left parenthesis G left parenthesis A 2 right parentheses labeled F left parenthesis G left parenthesis A 2 right parentheses. From element A to element F left parenthesis G left parenthesis A 2 right parentheses labeled left parenthesis F circle G right parenthesis left parenthesis A right parenthesis.

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Graphs of Functions – Appendix

- There are eight rows by eight columns of plotted points. There is a line passing through the points in the third column and a line passing through the points in the sixth row. The point in the first row fifth column, the point in the third row fourth column, and the point in the fifth row third column are shaded.

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Floor and Ceiling Functions – Appendix

- The X and Y axes range from -3 to 3, in increments of 1. There are horizontal segments of unit length. In the floor graph, each segment has a shaded point on its left end and a blank point on its right end. The segments are: from $x = -3$ to -2 and $y = -3$, from $x = -2$ to -1 and $y = -2$, from $x = -1$ to 0 and $y = -1$, from $x = 0$ to 1 and $y = 0$, from $x = 1$ to 2 and $y = 1$, from $x = 2$ to 3 and $y = 2$. In the ceiling graph, each segment has a blank point on its left end and a shaded point on its right end. The segments are: from $x = -3$ to -2 and $y = -2$, from $x = -2$ to -1 and $y = -1$, from $x = -1$ to 0 and $y = 0$, from $x = 0$ to 1 and $y = 1$, from $x = 1$ to 2 and $y = 2$, from $x = 2$ to 3 and $y = 3$.

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The Positive Rational Numbers are Countable – Appendix

- There are some rows and columns of elements. Each element is a fraction, where the numerator is a number of the row and the denominator is a number of the column. The elements are connected by arrows starting from the top left one. The path is as follows. One first circled, one half circled, two firsts circled. Three firsts circled, two halves not circled, one third circled. One fourth circled, two thirds circled, three halves circled. Four firsts circled, five firsts circled, four halves not circled. Three thirds not circled, two fourths not circled, one fifth circled, etc.

Jump to the image

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