Counting

Chapter 6

With Question/Answer Animations

Chapter Summary

The Basics of Counting

The Pigeonhole Principle

Permutations and Combinations

Binomial Coefficients and Identities

Generalized Permutations and Combinations

The Basics of Counting

Section 6.1

Section Summary

The Product Rule

The Sum Rule

The Subtraction Rule

The Division Rule

Examples, Examples, and Examples

Tree Diagrams

Basic Counting Principles: The Product Rule

The Product Rule: A procedure can be broken down into a sequence of two tasks. There are n_1 ways to do the first task and n_2 ways to do the second task. Then there are $n_1 \cdot n_2$ ways to do the procedure.

Example: How many bit strings of length seven are there?

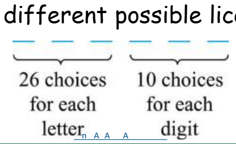
Solution: Since each of the seven bits is either a 0 or a 1, the answer is $2^7 = 128$.

The Product Rule

Example: How many different license plates can be made if each plate contains a sequence of three uppercase English letters followed by three digits?

Solution: By the product rule,

there are $26 \cdot 26 \cdot 26 \cdot 10 \cdot 10 \cdot 10 =$ 17,576,000 different possible license plates.



Counting Functions

Counting Functions: How many functions are there from a set with m elements to a set with n elements?

Solution: Since a function represents a choice of one of the n elements of the codomain for each of the m elements in the domain, the product rule tells us that there are $n \cdot n \cdots n = n^m$ such functions.

Counting One-to-One Functions: How many one-to-one functions are there from a set with m elements to one with n elements?

Solution: Suppose the elements in the domain are $a_1, a_2, \ldots, a_m$. There are n ways to choose the value of a_1 and n−1 ways to choose a_2, etc. The product rule tells us that there are $n(n-1)(n-2)\cdots(n-m+1)$ such functions.

Telephone Numbering Plan

Example: The North American numbering plan (NANP) specifies that a telephone number consists of 10 digits, consisting of a three-digit area code, a three-digit office code, and a four-digit station code. There are some restrictions on the digits.

- Let X denote a digit from 0 through 9.
- Let N denote a digit from 2 through 9.
- Let Y denote a digit that is 0 or 1.
- In the old plan (in use in the 1960s) the format was NYX-NNX-XXXX.
- In the new plan, the format is NXX-NXX-XXXX.

How many different telephone numbers are possible under the old plan and the new plan?

Solution: Use the Product Rule.

- There are $8 \cdot 2 \cdot 10 = 160$ area codes with the format NYX.
- There are $8 \cdot 10 \cdot 10 = 800$ area codes with the format NXX.
- There are $8 \cdot 8 \cdot 10 = 640$ office codes with the format NNX.
- There are $10 \cdot 10 \cdot 10 \cdot 10 = 10{,}000$ station codes with the format XXXX.

Number of old plan telephone numbers: $160 \cdot 640 \cdot 10{,}000 = 1{,}024{,}000{,}000$.

Number of new plan telephone numbers: $800 \cdot 800 \cdot 10{,}000 = 6{,}400{,}000{,}000$.

Counting Subsets of a Finite Set

Counting Subsets of a Finite Set: Use the product rule to show that the number of different subsets of a finite set S is $2^{|S|}$. (In Section 5.1, mathematical induction was used to prove this same result.)

Solution: When the elements of S are listed in an arbitrary order, there is a one-to-one correspondence between subsets of S and bit strings of length |S|. When the i^{th} element is in the subset, the bit string has a 1 in the i^{th} position and a 0 otherwise.

By the product rule, there are $2^{|S|}$ such bit strings, and therefore $2^{|S|}$ subsets.

Product Rule in Terms of Sets

If $A_1, A_2, \ldots, A_m$ are finite sets, then the number of elements in the Cartesian product of these sets is the product of the number of elements of each set.

The task of choosing an element in the Cartesian product $A_1 \times A_2 \times \cdots \times A_m$ is done by choosing an element in A_1, an element in A_2, …, and an element in A_m.

By the product rule, it follows that:

$$|A_1 \times A_2 \times \cdots \times A_m| = |A_1| \cdot |A_2| \cdot \cdots \cdot |A_m|$$

DNA and Genomes

A gene is a segment of a DNA molecule that encodes a particular protein; the entirety of genetic information of an organism is called its genome.

DNA molecules consist of two strands of blocks known as nucleotides. Each nucleotide is composed of bases: adenine (A), cytosine (C), guanine (G), or thymine (T).

The DNA of bacteria has between 10^5 and 10^7 links (one of the four bases). Mammals have between 10^8 and 10^{10} links. So, by the product rule there are at least 4^{10^5} different sequences of bases in the DNA of bacteria and 4^{10^8} different sequences of bases in the DNA of mammals.

The human genome includes approximately 23,000 genes, each with 1,000 or more links.

Biologists, mathematicians, and computer scientists all work on determining the DNA sequence (genome) of different organisms.

Basic Counting Principles: The Sum Rule

The Sum Rule: If a task can be done either in one of n_1 ways or in one of n_2, where none of the set of n_1 ways is the same as any of the n_2 ways, then there are $n_1 + n_2$ ways to do the task.

Example: The mathematics department must choose either a student or a faculty member as a representative for a university committee. How many choices are there for this representative if there are 37 members of the mathematics faculty and 83 mathematics majors and no one is both a faculty member and a student.

Solution: By the sum rule it follows that there are $37 + 83 = 120$ possible ways to pick a representative.

The Sum Rule for Sets

The sum rule can be phrased in terms of sets.

|A ∪ B|= |A| + |B| as long as A and B are disjoint sets. Or more generally,

$$|A_1 \cup A_2 \cup \cdots \cup A_m| = |A_1| + |A_2| + \cdots + |A_m|$$
$$\text{when } A_i \cap A_j = \varnothing \text{ for all } i, j.$$

The case where the sets have elements in common will be discussed when we consider the subtraction rule and taken up fully in Chapter 8.

Combining the Sum and Product Rules

Example: Suppose variable names in a programming language can be either a single letter or a letter followed by a digit. Find the number of possible variable names.

Solution: Use the product rule.
$26 + 26 \cdot 10 = 286$

Counting Passwords

Combining the sum and product rule allows us to solve more complex problems.
Example: Each user on a computer system has a password, which is six to eight characters long, where each character is an uppercase letter or a digit. Each password must contain at least one digit. How many possible passwords are there?

Solution: Let P be the total number of passwords, and let P_6, P_7, and P_8 be the passwords of length 6, 7, and 8.

- By the sum rule $P = P_6 + P_7 + P_8$.
- To find each of P_6, P_7, and P_8, we find the number of passwords of the specified length composed of letters and digits and subtract the number composed only of letters. We find that:
  $P_6 = 36^6 - 26^6 = 2{,}176{,}782{,}336 - 308{,}915{,}776 = 1{,}867{,}866{,}560.$
  $P_7 = 36^7 - 26^7 = 78{,}364{,}164{,}096 - 8{,}031{,}810{,}176 = 70{,}332{,}353{,}920.$
  $P_8 = 36^8 - 26^8 = 2{,}821{,}109{,}907{,}456 - 208{,}827{,}064{,}576 = 2{,}612{,}282{,}842{,}880.$

Consequently, $P = P_6 + P_7 + P_8 = 2{,}684{,}483{,}063{,}360.$

Internet Addresses

Version 4 of the Internet Protocol (IPv4) uses 32 bits.

Bit Number	0	1	2	3	4	8	16	24	31
Class A	0	netid					hostid		
Class B	1	0	netid					hostid	
Class C	1	1	0	netid					hostid
Class D	1	1	1	0	Multicast Address				
Class E	1	1	1	1	0	Address			

Class A Addresses: used for the largest networks, a 0,followed by a 7-bit netid and a 24-bit hostid.
Class B Addresses: used for the medium-sized networks, a 10,followed by a 14-bit netid and a 16-bit hostid.
Class C Addresses: used for the smallest networks, a 110, followed by a 21-bit netid and a 8-bit hostid.

- Neither Class D nor Class E addresses are assigned as the address of a computer on the internet. Only Classes A, B, and C are available.
- 1111111 is not available as the netid of a Class A network.
- Hostids consisting of all 0s and all 1s are not available in any network.

Counting Internet Addresses

Example: How many different IPv4 addresses are available for computers on the internet?

Solution: Use both the sum and the product rule. Let x be the number of available addresses, and let x_A, x_B, and x_C denote the number of addresses for the respective classes.

- To find x_A: $2^7 - 1 = 127$ netids; $2^{24} - 2 = 16{,}777{,}214$ hostids
  $x_A = 127 \cdot 16{,}777{,}214 = 2{,}130{,}706{,}178$
- To find x_B: $2^{14} = 16{,}384$ netids; $2^{16} - 2 = 16{,}534$ hostids
  $x_B = 16{,}384 \cdot 16{,}534 = 1{,}073{,}709{,}056$
- To find x_C: $2^{21} = 2{,}097{,}152$ netids; $2^8 - 2 = 254$ hostids
  $x_C = 2{,}097{,}152 \cdot 254 = 532{,}676{,}608$
- Hence, the total number of available IPv4 addresses is
  $x = x_A + x_B + x_C$
  $= 2{,}130{,}706{,}178 + 1{,}073{,}709{,}056 + 532{,}676{,}608$
  $= 3{,}737{,}091{,}842$

Not Enough Today! The newer IPv6 protocol solves the problem of too few addresses.

Basic Counting Principles: Subtraction Rule

Subtraction Rule: If a task can be done either in one of n_1 ways or in one of n_2 ways, then the total number of ways to do the task is $n_1 + n_2$ minus the number of ways to do the task that are common to the two different ways.

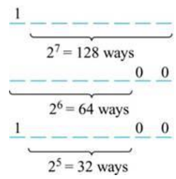
Also known as the principle of inclusion-exclusion:

$$|A \cup B| = |A| + |B| - |A \cap B|$$

Counting Bit Strings

Example: How many bit strings of length eight either start with a 1 bit or end with the two bits 00?

Solution: Use the subtraction rule.



- Number of bit strings of length eight that start with a 1 bit: $2^7 = 128$
- Number of bit strings of length eight that end with bits 00: $2^6 = 64$
- Number of bit strings of length eight that start with a 1 bit and end with bits 00 : $2^5 = 32$

Hence, the number is 128 + 64 − 32 = 160

Basic Counting Principles: Division Rule*

Division Rule: There are n/d ways to do a task if it can be done using a procedure that can be carried out in n ways, and for every way w, exactly d of the n ways correspond to way w.
Restated in terms of sets: If the finite set A is the union of n pairwise disjoint subsets each with d elements, then n = |A|/d.
In terms of functions: If f is a function from A to B, where both are finite sets, and for every value y ∈ B there are exactly d values x ∈ A such that f(x) = y, then |B| = |A|/d.

Example: How many ways are there to seat four people around a circular table, where two seatings are considered the same when each person has the same left and right neighbor?

Solution: Number the seats around the table from 1 to 4 proceeding clockwise. There are four ways to select the person for seat 1, 3 for seat 2, 2, for seat 3, and one way for seat 4. Thus there are 4! = 24 ways to order the four people. But since two seatings are the same when each person has the same left and right neighbor, for every choice for seat 1, we get the same seating.

Therefore, by the division rule, there are 24/4 = 6 different seating arrangements.

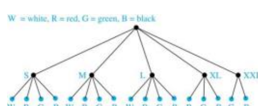
Tree Diagrams

Tree Diagrams: We can solve many counting problems through the use of tree diagrams, where a branch represents a possible choice and the leaves represent possible outcomes.

Example: Suppose that "I Love Discrete Math" T-shirts come in five different sizes: S, M, L, XL, and XXL. Each size comes in four colors (white, red, green, and black), except XL, which comes only in red, green, and black, and XXL, which comes only in green and black. What is the minimum number of shirts that the campus bookstore needs to stock to have one of each size and color available?

Solution: Draw the tree diagram.

The store must stock 17 T-shirts.



The Pigeonhole Principle

Section 6.2

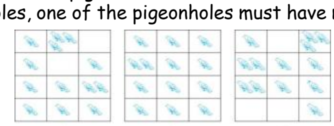
Section Summary

The Pigeonhole Principle

The Generalized Pigeonhole Principle

The Pigeonhole Principle

If a flock of 20 pigeons roosts in a set of 19 pigeonholes, one of the pigeonholes must have more than 1 pigeon.



Pigeonhole Principle: If k is a positive integer and k + 1 objects are placed into k boxes, then at least one box contains two or more objects.
Proof: We use a proof by contradiction. Suppose none of the k boxes has more than one object. Then the total number of objects would be at most k. This contradicts the statement that we have k + 1 objects.

The Pigeonhole Principle

Corollary 1: A function f from a set with k + 1 elements to a set with k elements is not one-to-one.

Proof: Use the pigeonhole principle.

- Create a box for each element y in the codomain of f.
- Put in the box for y all of the elements x from the domain such that f(x) = y.
- Because there are k + 1 elements and only k boxes, at least one box has two or more elements.

Hence, f can't be one-to-one.

Pigeonhole Principle

Example: Among any group of 367 people, there must be at least two with the same birthday, because there are only 366 possible birthdays.

Example: Show that for every integer n, there is a multiple of n that has only 0s and 1s in its decimal expansion.

Solution: Let n be a positive integer. Consider the n + 1 integers 1, 11, 111, …., 11…1 (where the last has n + 1 1s). There are n possible remainders when an integer is divided by n. By the pigeonhole principle, when each of the n + 1 integers is divided by n, at least two must have the same remainder. Subtract the smaller from the larger and the result is a multiple of n that has only 0s and 1s in its decimal expansion.

The Generalized Pigeonhole Principle

The Generalized Pigeonhole Principle: If N objects are placed into k boxes, then there is at least one box containing at least ⌈N/k⌉ objects.
Proof: We use a proof by contradiction. Suppose that none of the boxes contains more than ⌈N/k⌉ − 1 objects. Then the total number of objects is at most

$$k\left(\left\lceil \frac{N}{k} \right\rceil - 1\right) < k\left(\left(\frac{N}{k} + 1\right) - 1\right) = N,$$

where the inequality ⌈N/k⌉ < ⌈N/k⌉ + 1 has been used. This is a contradiction because there are a total of n objects.

Example: Among 100 people there are at least ⌈100/12⌉ = 9 who were born in the same month.

The Generalized Pigeonhole Principle

Example: a) How many cards must be selected from a standard deck of 52 cards to guarantee that at least three cards of the same suit are chosen?

b) How many must be selected to guarantee that at least three hearts are selected?

Solution: a) We assume four boxes; one for each suit. Using the generalized pigeonhole principle, at least one box contains at least ⌈N/4⌉ cards. At least three cards of one suit are selected if ⌈N/4⌉ ≥ 3. The smallest integer N such that ⌈N/4⌉ ≥3 is N = 2 ∙ 4 + 1 = 9.

b) A deck contains 13 hearts and 39 cards which are not hearts. So, if we select 41 cards, we may have 39 cards which are not hearts along with 2 hearts. However, when we select 42 cards, we must have at least three hearts. (Note that the generalized pigeonhole principle is not used here.)

Permutations and Combinations

Section 6.3

Section Summary

Permutations

Combinations

Combinatorial Proofs

Permutations

Definition: A permutation of a set of distinct objects is an ordered arrangement of these objects. An ordered arrangement of r elements of a set is called an r-permutation.

Example: Let S = {1,2,3}.

- The ordered arrangement 3,1,2 is a permutation of S.
- The ordered arrangement 3,2 is a 2-permutation of S.

The number of r-permutations of a set with n elements is denoted by P(n,r).

- The 2-permutations of S = {1,2,3} are 1,2; 1,3; 2,1; 2,3; 3,1; and 3,2. Hence, P(3,2) = 6.

A Formula for the Number of Permutations

Combinations

Theorem 2: The number of r-combinations of a set with n elements, where $n \geq r \geq 0$, equals

$$C(n,r) = \frac{n!}{(n-r)!r!}.$$

Proof: By the product rule $P(n, r) = C(n,r) \cdot P(r,r)$. Therefore,

$$C(n,r) = \frac{p(n,r)}{P(r,r)} = \frac{n!/(n-r)!}{r!/(r-r)!} = \frac{n!}{(n-r)!r!}.$$

Combinations

Example: How many poker hands of five cards can be dealt from a standard deck of 52 cards? Also, how many ways are there to select 47 cards from a deck of 52 cards?

Solution: Since the order in which the cards are dealt does not matter, the number of five card hands is:

$$C(52,5) = \frac{52!}{5!47!}$$
$$= \frac{52 \cdot 51 \cdot 50 \cdot 49 \cdot 48}{5 \cdot 4 \cdot 3 \cdot 2 \cdot 1} = 26 \cdot 17 \cdot 10 \cdot 49 \cdot 12 = 2{,}598{,}960$$

The different ways to select 47 cards from 52 is

$$C(52,47) = \frac{52!}{47!5!} = C(52,5) = 2{,}598{,}960$$

This is a special case of a general result.

Combinations

Corollary 2: Let n and r be nonnegative integers with $r \leq n$. Then $C(n, r) = C(n, n - r)$.

Proof: From Theorem 2, it follows that

$$C(n,r) = \frac{n!}{(n-r)!r!}$$

and

$$C(n,n-r) = \frac{n!}{(n-r)![n-(n-r)]!} = \frac{n!}{(n-r)!r!}.$$

- Hence, $C(n, r) = C(n, n - r)$.

This result can be proved without using algebraic manipulation. →

Combinations

Example: How many ways are there to select five players from a 10-member tennis team to make a trip to a match at another school.

Solution: By Theorem 2, the number of combinations is

$$C(10,5) = \frac{10!}{5!5!} = 252.$$

Example: A group of 30 people have been trained as astronauts to go on the first mission to Mars. How many ways are there to select a crew of six people?

Solution: By Theorem 2, the number of possible crews is

$$C(30,6) \quad \frac{30!}{6!24!} \quad \frac{30 \cdot 29 \cdot 28 \cdot 27 \cdot 26 \cdot 25}{6 \;\; 5 \;\; 4 \;\; 3 \;\; 2 \;\; 1} \quad 593{,}775$$

Binomial Coefficients and Identities

Section 6.4

Section Summary

The Binomial Theorem

Pascal's Identity and Triangle

Powers of Binomial Expressions

Definition: A *binomial* expression is the sum of two terms, such as $x + y$. (More generally, these terms can be products of constants and variables.)

- We can use counting principles to find the coefficients in the expansion of $(x + y)^n$ where n is a positive integer.
- To illustrate this idea, we first look at the process of expanding $(x + y)^3$.
- $(x + y)(x + y)(x + y)$ expands into a sum of terms that are the product of a term from each of the three sums.
- Terms of the form x^3, x^2y, xy^2, y^3 arise. The question is what are the coefficients?
  - To obtain x^3, an x must be chosen from each of the sums. There is only one way to do this. So, the coefficient of x^3 is 1.
  - To obtain x^2y, an x must be chosen from two of the sums and a y from the other. There are $\binom{3}{2}$ ways to do this and so the coefficient of x^2y is 3.
- To obtain xy^2, an x must be chosen from of the sums and a y from the other two. There are $\binom{3}{1}$ ways to do this and so the coefficient of xy^2 is 3.
- To obtain y^3, a y must be chosen from each of the sums. There is only one way to do this. So, the coefficient of y^3 is 1.

We have used a counting argument to show that $(x + y)^3 = x^3 + 3x^2y + 3xy^2 + y^3$.
Next we present the binomial theorem gives the coefficients of the terms in the expansion of $(x + y)^n$.

Binomial Theorem

Binomial Theorem: Let x and y be variables, and n a nonnegative integer. Then:

$$(x+y)^n = \sum_{j=0}^{n}\binom{n}{j}x^{n-j}y^j = \binom{n}{0}x^n + \binom{n}{1}x^{n-1}y + \cdots + \binom{n}{n-1}xy^{n-1} + \binom{n}{n}y^n.$$

- **Proof**: We use combinatorial reasoning . The terms in the expansion of $(x + y)^n$ are of the form $x^{n-j}y^j$ for j = 0,1,2,…,n. To form the term $x^{n-j}y^j$, it is necessary to choose $n-j$ xs from the n sums. Therefore, the coefficient of $x^{n-j}y^j$ is $\binom{n}{n-j}$ which equals $\binom{n}{j}$.

Using the Binomial Theorem

Example: What is the coefficient of $x^{12}y^{13}$ in the expansion of $(2x - 3y)^{25}$?

Solution: We view the expression as $(2x + (-3y))^{25}$. By the binomial theorem

$$(2x + (-3y))^{25} = \sum_{j=0}^{25}\binom{25}{j}2x^{25-j}(-3y)^j.$$

Consequently, the coefficient of $x^{12}y^{13}$ in the expansion is obtained when j = 13.

$$\binom{25}{13}2^{12}(-3)^{13} = \frac{25!}{13!12!}2^{12}3^{13}.$$

A Useful Identity

Corollary 1: With $n \geq 0$, $\sum_{k=0}^{n}\binom{n}{k} = 2^n$.

Proof (*using binomial theorem*): With x = 1 and y = 1, from the binomial theorem we see that:

$$2^n = (1+1)^n = \sum_{k=0}^{n}\binom{n}{k}1^k1^{(n-k)} = \sum_{k=0}^{n}\binom{n}{k}.$$

Proof (*combinatorial*): Consider the subsets of a set with n elements. There are $\binom{n}{0}$ subsets with zero elements, $\binom{n}{1}$ with one element, $\binom{n}{2}$ with two elements, …, and $\binom{n}{n}$ with n elements. Therefore the total is $\sum_{k=0}^{n}\binom{n}{k}$.

Since, we know that a set with n elements has 2^n subsets, we conclude: $\sum_{k=0}^{n}\binom{n}{k} = 2^n$.

Pascal's Identity

Blaise Pascal (1623-1662)

Pascal's Identity: If n and k are integers with $n \geq k \geq 0$, then $\binom{n+1}{k} = \binom{n}{k-1} + \binom{n}{k}$.

Proof (*combinatorial*): Let T be a set where $|T| = n + 1$, $a \in T$, and $S = T - \{a\}$. There are $\binom{n+1}{k}$ subsets of T containing k elements.

Each of these subsets either:

- contains a with $k - 1$ other elements, or
- contains k elements of S and not a.

There are

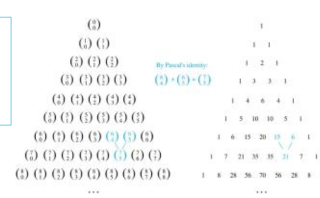
- $\binom{n}{k-1}$ subsets of k elements that contain a, since there are $\binom{n}{k-1}$ subsets of $k - 1$ elements of S,
- $\binom{n}{k}$ subsets of k elements of T that do not contain a, since there are $\binom{n}{k}$ subsets of k elements of S.

Hence, $\binom{n+1}{k} = \binom{n}{k-1} + \binom{n}{k}$

A

The nth row in the triangle consists of the binomial coefficients

$\binom{n}{k}$, k = 0,1,….,n.



By Pascal's identity, adding two adjacent binomial coefficients results is the binomial coefficient in the next row between these two coefficients.

Generalized Permutations and Combinations

Section 6.5

Section Summary

Permutations with Repetition

Combinations with Repetition

Permutations with Indistinguishable Objects

Distributing Objects into Boxes

Permutations with Repetition

Theorem 1: The number of r-permutations of a set of n objects with repetition allowed is n^r.

Proof: There are n ways to select an element of the set for each of the r positions in the r-permutation when repetition is allowed. Hence, by the product rule there are n^r r-permutations with repetition.

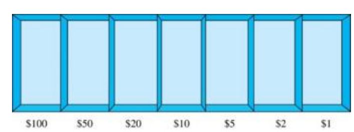
Example: How many strings of length r can be formed from the uppercase letters of the English alphabet?

Solution: The number of such strings is 26^r, which is the number of r-permutations of a set with 26 elements.

Combinations with Repetition

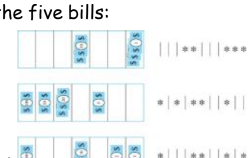
Example: How many ways are there to select five bills from a box containing at least five of each of the following denominations: $1, $2, $5, $10, $20, $50, and $100?

Solution: Place the selected bills in the appropriate position of a cash box illustrated below:



Combinations with Repetition

Some possible ways of placing the five bills:



The number of ways to select five bills corresponds to the number of ways to arrange six bars and five stars in a row.

This is the number of unordered selections of 5 objects from a set of 11. Hence, there are

$$C(11,5) = \frac{11!}{5!6!} = 462$$

ways to choose five bills with seven types of bills.

Combinations with Repetition

Theorem 2: The number of r-combinations from a set with n elements when repetition of elements is allowed is

$$C(n+r \quad 2\,r) = C(n+r \quad 2\,n \quad 2)/$$

Proof: Each r-combination of a set with n elements with repetition allowed can be represented by a list of n –1 bars and r stars. The bars mark the n cells containing a star for each time the ith element of the set occurs in the combination.

The number of such lists is C(n + r – 1, r), because each list is a choice of the r positions to place the stars, from the total of n + r – 1 positions to place the stars and the bars. This is also equal to C(n + r – 1, n –1), which is the number of ways to place the n –1 bars.

Combinations with Repetition

Example: How many solutions does the equation

$$x_1 + x_2 + x_3 = 11$$

have, where x_1, x_2, and x_3 are nonnegative integers?

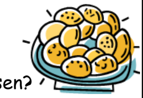
Solution: Each solution corresponds to a way to select 11 items from a set with three elements: x_1 elements of type one, x_2 of type two, and x_3 of type three.

By Theorem 2 it follows that there are

$$C(3+11-1,11) = C(13,11) = C(13,2) = \frac{13\cdot 2}{1\cdot 2} = 78$$

solutions.

Combinations with Repetition



Example: Suppose that a cookie shop has four different kinds of cookies. How many different ways can six cookies be chosen?

Solution: The number of ways to choose six cookies is the number of 6-combinations of a set with four elements. By Theorem 2

$$C(9,6) = C(9,3) = \frac{9\cdot 8\cdot 7}{1\cdot 2\cdot 3} = 84$$

is the number of ways to choose six cookies from the four kinds.

Summarizing the Formulas for Counting Permutations and Combinations with and without Repetition

TABLE 1 Combinations and Permutations With and Without Repetition.

Type	*Repetition Allowed?*	*Formula*
r-permutations	No	$\frac{n!}{(n-r)!}$
r-combinations	No	$\frac{n!}{r!(n-r)!}$
r-permutations	Yes	n^r
r-combinations	Yes	$\frac{(n+r-1)!}{r!(n-1)!}$

Permutations with Indistinguishable Objects

Example: How many different strings can be made by reordering the letters of the word SUCCESS.

Solution: There are seven possible positions for the three S's, two C's, one U, and one E.

- The three S's can be placed in *C*(7,3) different ways, leaving four positions free.
- The two *C*'s can be placed in *C*(4,2) different ways, leaving two positions free.
- The U can be placed in *C*(2,1) different ways, leaving one position free.
- The E can be placed in *C*(1,1) way.

By the product rule, the number of different strings is:

$$C(7,3)C(4,2)C(2,1)C(1,1) = \frac{7!}{3!4!}\cdot\frac{4!}{2!2!}\cdot\frac{2!}{1!1!}\cdot\frac{1!}{3!2!1!1!} = 420.$$

The reasoning can be generalized to the following theorem. →

Permutations with Indistinguishable Objects

Theorem 3: The number of different permutations of n objects, where there are n_1 indistinguishable objects of type 1, n_2 indistinguishable objects of type 2, ….., and n_k indistinguishable objects of type k, is:

$$\frac{n!}{n_1!n_2!\cdots n_k!}.$$

Proof: By the product rule the total number of permutations is:
$C(n, n_1)\, C(n - n_1, n_2) \cdots C(n - n_1 - n_2 - \cdots - n_k, n_k)$ since:

- The n_1 objects of type one can be placed in the n positions in $C(n, n_1)$ ways, leaving $n - n_1$ positions.
- Then the n_2 objects of type two can be placed in the $n - n_1$ positions in $C(n - n_1, n_2)$ ways, leaving $n - n_1 - n_2$ positions.
- Continue in this fashion, until n_k objects of type k are placed in $C(n - n_1 - n_2 - \cdots - n_k, n_k)$ ways.

The product can be manipulated into the desired result as follows:

$$\frac{n!}{n_1!(n-n_1)!}\,\frac{(n-n_1)!}{n_2!(n-n_1-n_2)!}\cdots\frac{(n-n_1-\cdots-n_{k-1})!}{n_k!0!} = \frac{n!}{n_1!n_2!\cdots n_k!}.$$

Appendix of Image Long Descriptions

The Product Rule – Appendix

There are 6 gaps, 3 for letters and 3 for digits. There is a curly bracket under the gaps for letters showing that there are 26 choices for each letter. Also, there is a curly bracket under the gaps for digits showing that there are 10 choices for each digit.

Internet Addresses – Appendix

There is a table consisting of 32 columns and 5 rows. The columns are numbered from 0 to 31, and they represent a bit number. The rows are Class A, Class B, Class C, Class D, and Class E. Bit 0 in Class A is 0. Bits 1 through 7 are labeled net ID. Bits 8 through 31 are labeled host ID. Bit 0 in Class B is 1, bit 1 is 0. Bits 2 through 15 are labeled net ID. Bits 16 through 31 are labeled host ID. Bits 0 and 1 in Class C are 1, bit 2 is 0, bits 3 through 23 are labeled net ID. Bits 24 through 31 are labeled host ID. Bits 0, 1, and 2 in Class D are 1, bit 3 is 0. Bits 4 through 31 are labeled Multicast Address. Bits 0, 1, 2, and 3 in Class D are 1. Bit 4 is 0. Bits 5 through 31 are labeled Address.

Counting Bit Strings – Appendix

There are three lines of 8 gaps each. In the first line, the first gap is 1, and there is a curly bracket under other gaps showing that there are 2 to the seventh power equal to 128 ways to place 0 and 1 on the remaining 7 gaps. The two last gaps of the second line are 0, and there is a curly bracket under other gaps showing that there are 2 to the sixth power equal to 64 ways to place 0 and 1 on the remaining 6 gaps. The first gap of the third line is 1, two last gaps are 0. Also, there is a curly bracket under other gaps showing that there are 2 to the fifth power equal to 32 ways to place 0 and 1 on the remaining 5 gaps.

Tree Diagrams – Appendix

The tree has a root on a basic level. The root has five branches to the vertices that labeled as S, M, L, X L, and X X L. Each of vertices S, M, and L has four branches to the vertices labeled W, R, G, and B. Vertex X L has three branches to vertices labeled R, G, and B. Vertex X X L has two branches to vertices labeled G and B.

The Pigeonhole Principle – Appendix

There are three 4 by 3 tables labeled A, B, and C. Table A. Cells 1, 3 and 2, 2 are empty. There are 3 pigeons in cell 1, 2 and 2 pigeons in cell 3, 3. All other cells have one pigeon. Table B. There are 2 pigeons in cell 1,1. All other cells have one pigeon. Table C. Cells 1, 2 and 4, 1 and 4, 2 are empty. There are 3 pigeons in cell 1, 3. There are 2 pigeons in cells 2, 1 and 3, 1. All other cells have one pigeon.

Pascal's Triangle – Appendix

The elements of the first triangle are binomial coefficients. The top number is the row number, and the bottom number is the column number, both starting from 0. Coefficient 0, 0 is at the top. The elements of the second triangle are natural numbers. 1 is at the top of the triangle and along the left and right edges. The numbers between them are the sums of two numbers above.

Combinations with Repetition – Appendix

The first way is 2 10 dollar bills from the fourth compartment and 3 1 dollar bills from the seventh compartment. Or 3 bars, 2 stars, 3 bars, and 3 stars. The second way is 1 100 dollar bill from the first compartment. 1 50 dollar bill from the second compartment. 2 20 dollar bills from the third compartment, and 1 5 dollar bill from the fifth compartment. Or 1 star, 1 bar, 1 star. 1 bar, 2 stars, 2 bars. 1 star, and 2 bars. The third way is 1 100 dollar bill from the first compartment. 2 10 dollar bills from the fourth compartment. 1 2 dollar bill from the sixth compartment, and 1 1 dollar bill from the seventh compartment. Or 1 star, 3 bars, 2 stars. 2 bars, 1 star, 1 bar, and 1 star.