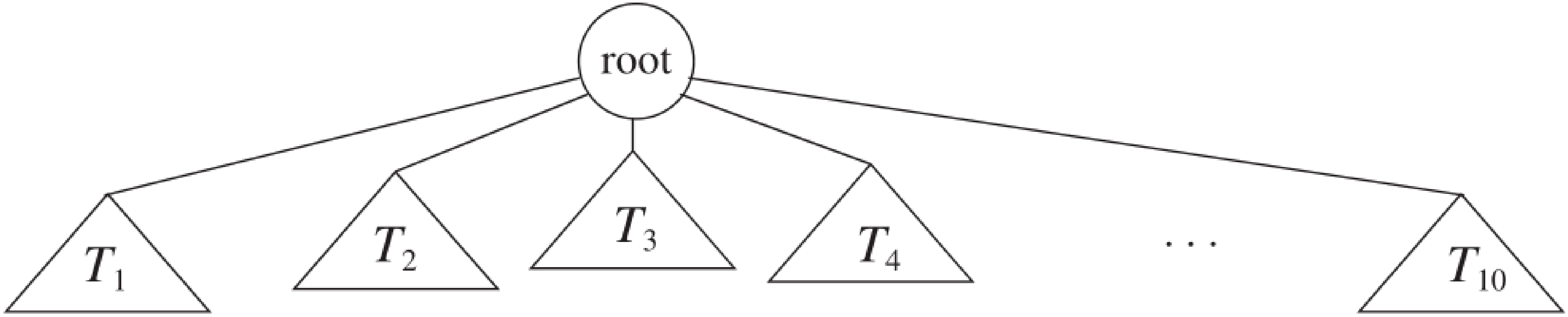
Chapter 4
Trees

Introduction

- for large input, even linear access time may be prohibitive
  - we need data structures that exhibit average running times closer to $O(\log N)$
  - binary search tree

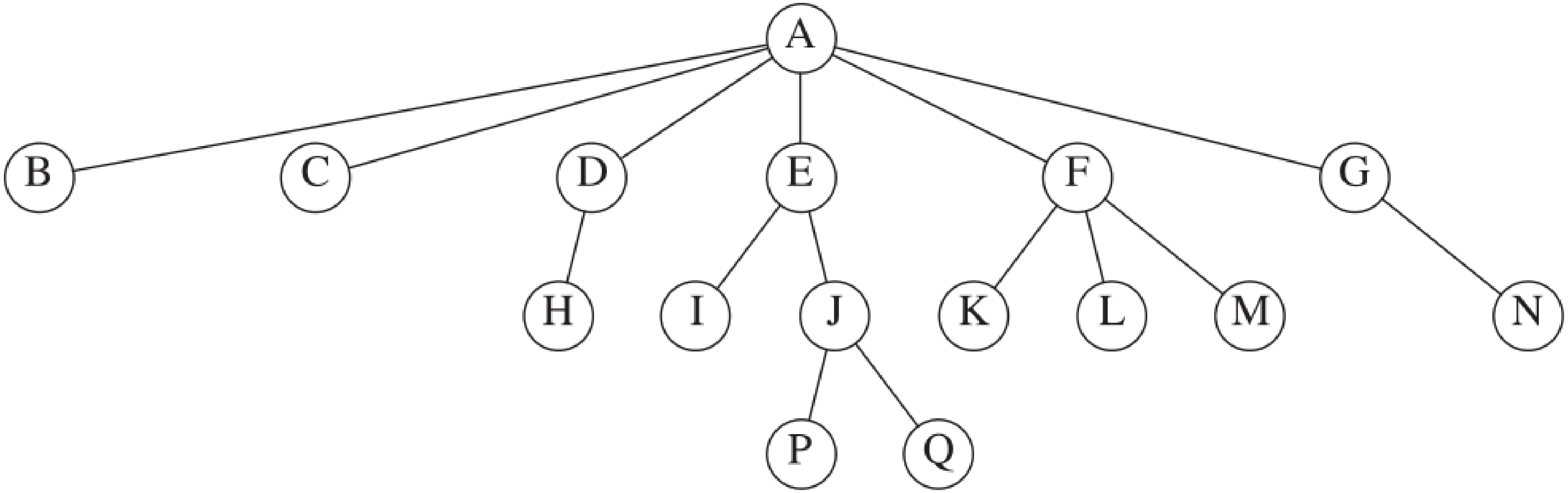
Terminology

- recursive definition of **tree**
  - collection of nodes (may be empty)
  - distinguished node, *r*, is the **root**
  - zero or more nonempty subtrees T_1, T_2, …T_k, each of whose roots are connected by a directed **edge** from *r*
- root of each subtree is a **child** of *r*
- *r* is the **parent** of each subtree
- tree of N nodes has $N - 1$ edges



Terminology

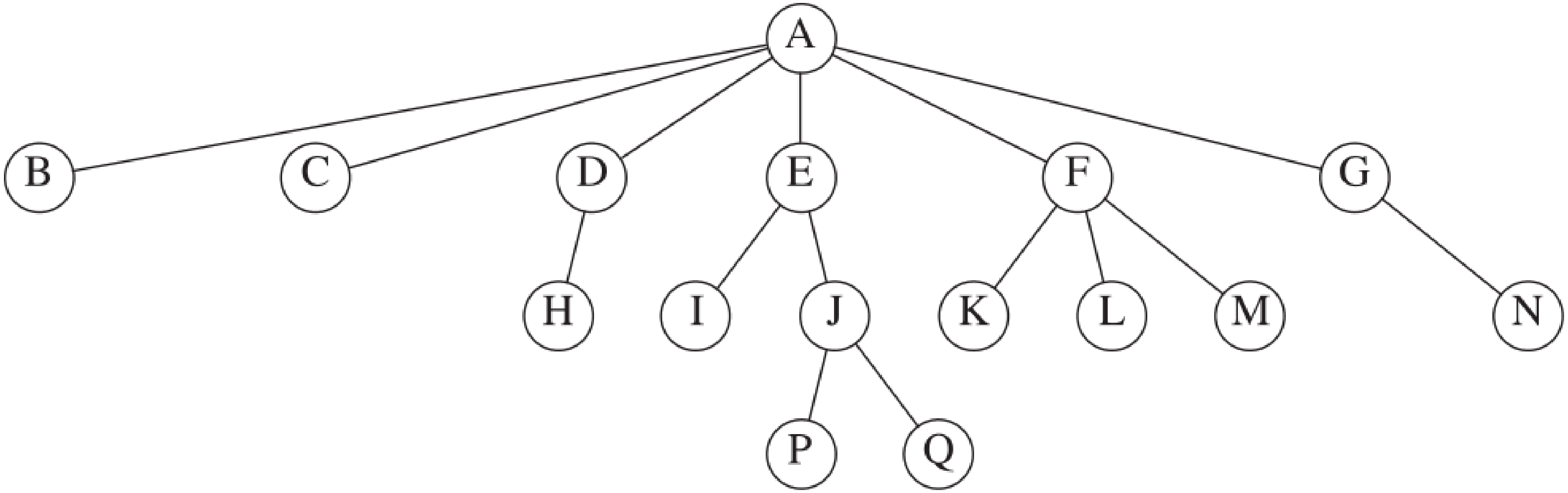
- example tree



- nodes with no children are called **leaves** (e.g., B, C, H, I, P, Q, K, L, M, N)
- nodes with the same parent are **siblings** (e.g., K, L, M)
- **parent**, **grandparent**, **grandchild, ancestor, descendant, proper ancestor, proper descendant**

Terminology

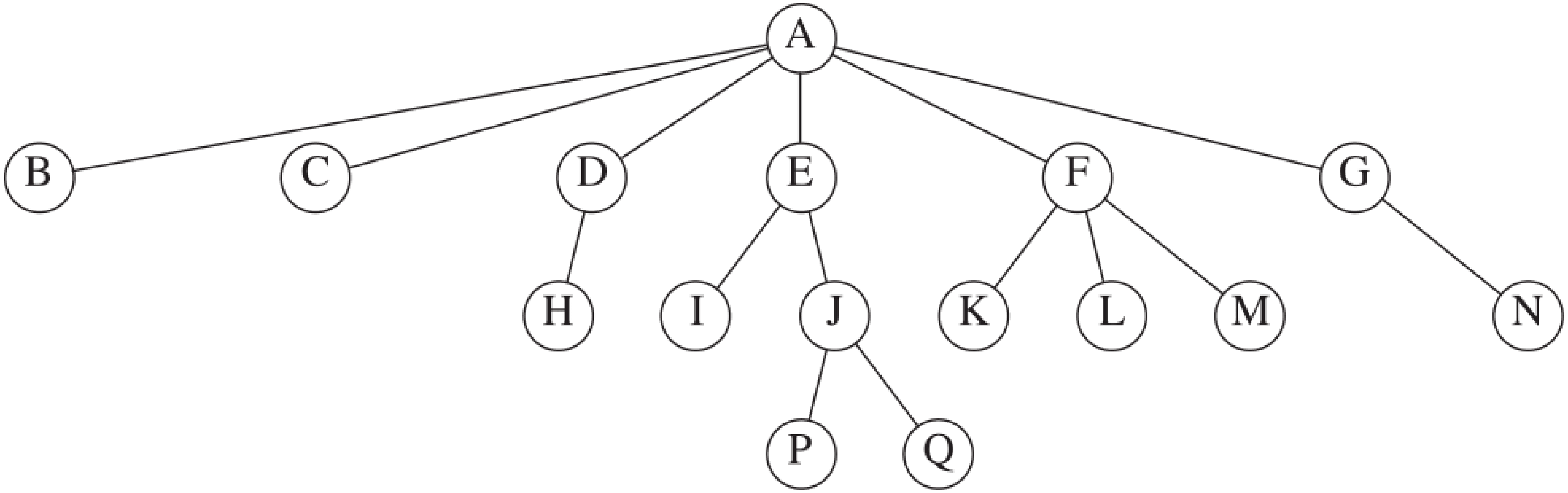
- example tree



- **path** from n_1 to n_k is a sequence of nodes $n_1, n_2, \ldots, n_k$ where n_1 is the parent of n_{i+1} for $1 <= i < k$
- **length** of path is number of edges on path ($k – 1$)
  - path of length 0 from every node to itself
  - exactly one path from the root to each node

Terminology

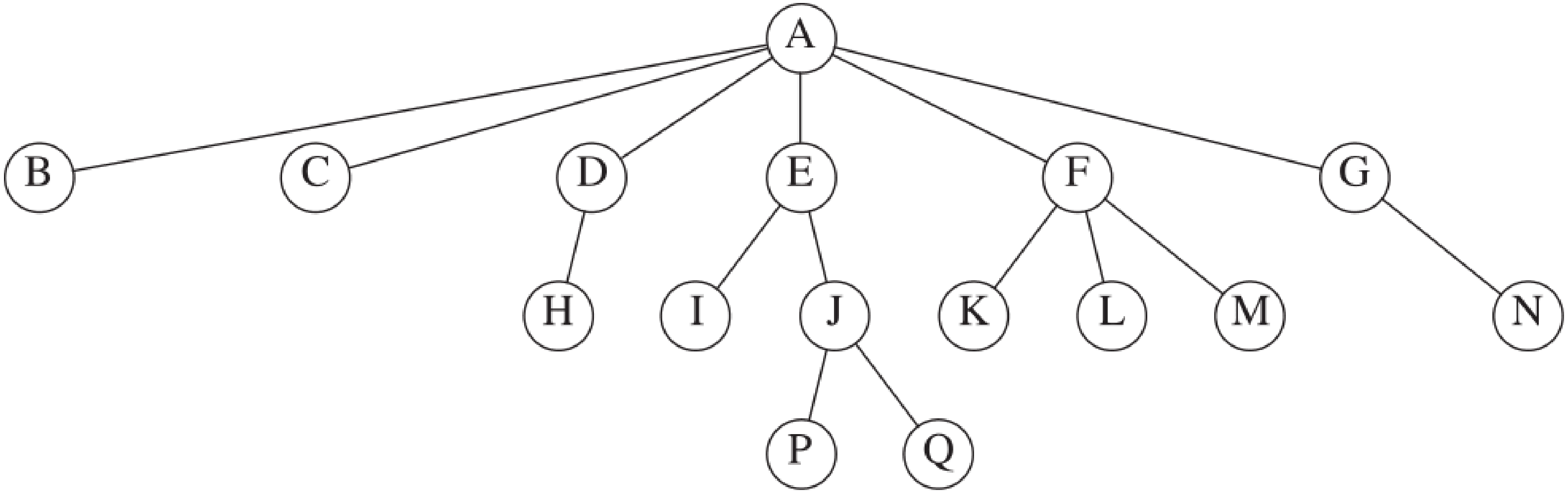
- example tree



- **depth** from n_i is the length of the unique path from the root to n_i
  - root is at depth 0
- **height** of n_i is the length of the longest path from n_i to a leaf
  - all leaves at height 0
  - height of the tree is equal to the height of the root

Terminology

- example tree

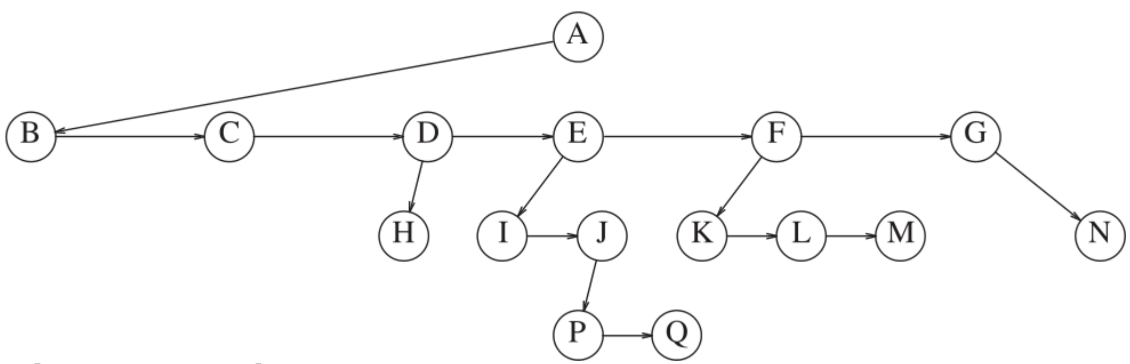


- E is at depth 1 and height 2
- F is at depth 1 and height 1
- depth of tree is 3

Implementation of Trees

- each node could have <u>data</u> and a link to each <u>child</u>
  - number of children is unknown and may be large, which could lead to <u>wasted space</u>
  - instead, keep children in a linked list

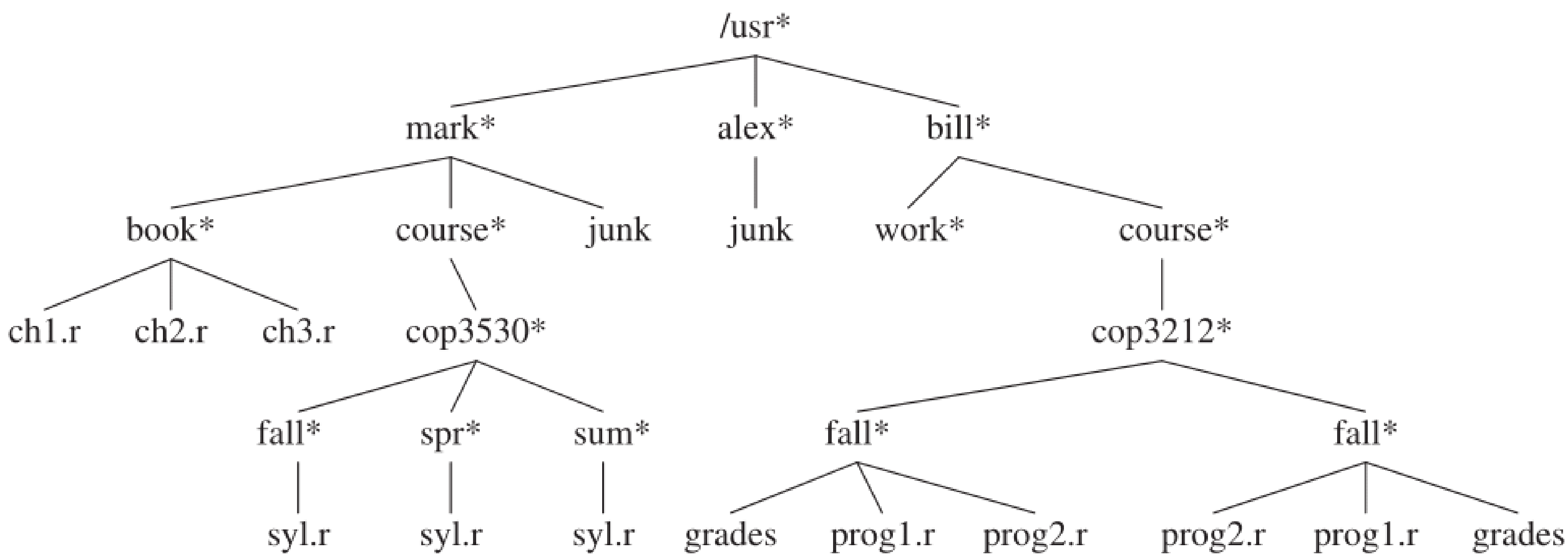
```
struct TreeNode
{
    Object    element;
    TreeNode *firstChild;
    TreeNode *nextSibling;
};
```



  - null links not shown

Tree Traversals with Application

- many <u>applications</u> for trees
  - subdirectory structure in Unix
  - pathname built into tree



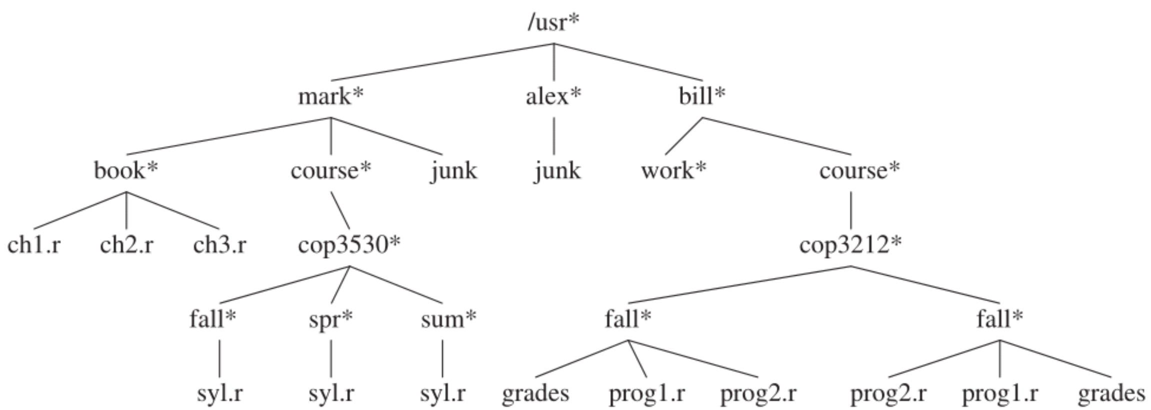
Tree Traversals with Application

- goal: list all files in a directory
  - depth denoted by tabs
  - begins at root

```
void FileSystem::listAll( int depth = 0 ) const
{
    printName( depth );  // Print the name of the object
    if( isDirectory( ) )
        for each file c in this directory (for each child)
            c.listAll( depth + 1 );
}
```

Tree Traversals with Application

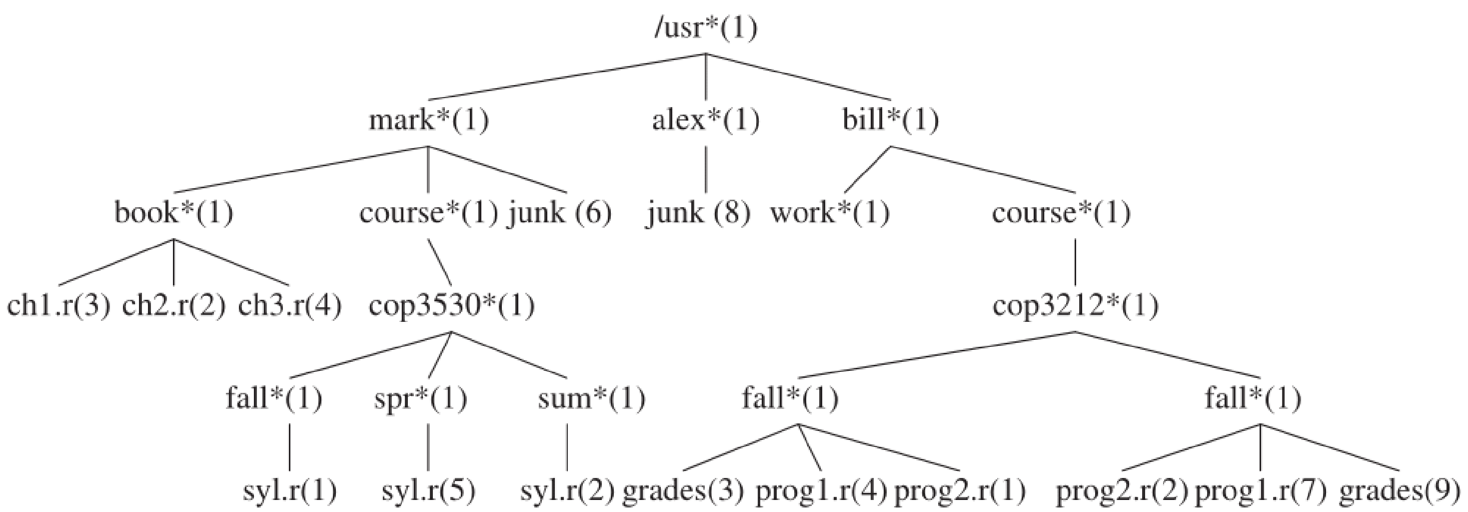
- code prints directories/files in <u>preorder</u> traversal
  - runs in $O(N)$




```
/usr
    mark
        book
            ch1.r
            ch2.r
            ch3.r
        course
            cop3530
                fall
                    syl.r
                spr
                    syl.r
                sum
                    syl.r
        junk
    alex
        junk
    bill
        work
        course
            cop3212
                fall
                    grades
                    prog1.r
                    prog2.r
                fall
                    prog2.r
                    prog1.r
                    grades
```

Tree Traversals with Application

– for postorder traversal, numbers in parentheses represent the number of disk blocks for each file



Tree Traversals with Application

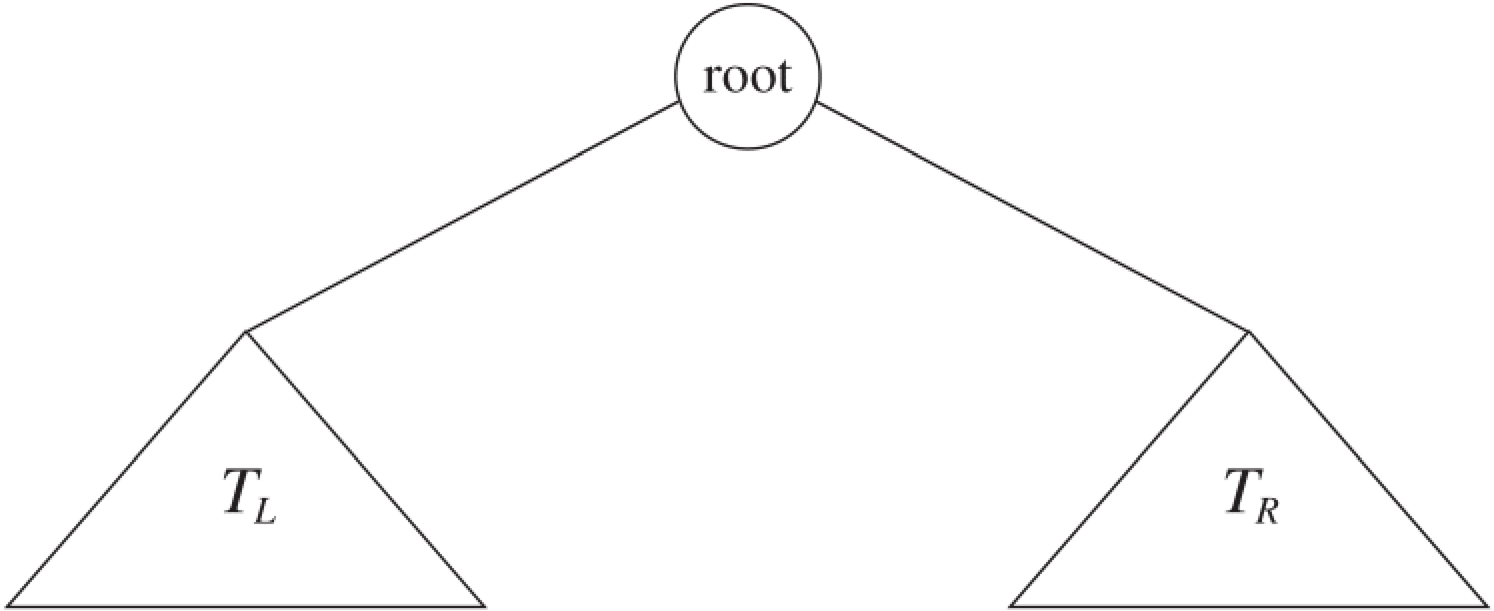
- size method to find number of <u>blocks</u> for each file
  - directories use 1 block of space

```
int FileSystem::size( ) const
{
    int totalSize = sizeOfThisFile( );

    if( isDirectory( ) )
        for each file c in this directory (for each child)
            totalSize += c.size( );

     return totalSize;
}
```

```
            ch1.r          3
            ch2.r          2
            ch3.r          4
        book              10
                    syl.r  1
                fall       2
                    syl.r  5
                spr        6
                    syl.r  2
                sum        3
            cop3530       12
        course            13
        junk               6
    mark                  30
        junk               8
    alex                   9
        work               1
                    grades 3
                    prog1.r 4
                    prog2.r 1
                fall       9
                    prog2.r 2
                    prog1.r 7
                    grades 9
                fall      19
            cop3212       29
        course            30
    bill                  32
/usr                      72
```

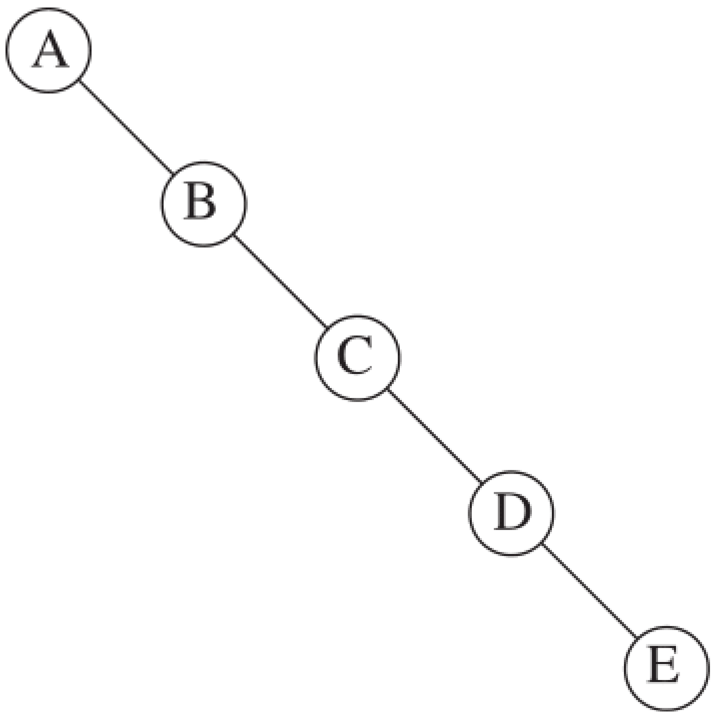
Binary Trees

- in binary trees, nodes can have no more than <u>two</u> children
- binary tree below consists of a root and two subtrees, T_L and T_R, both of which could possibly be <u>empty</u>



Binary Trees

- depth of a binary tree is considerably smaller than <u>N</u>
- average depth is $O(\sqrt{N})$
- average depth for a binary search tree is <u>$O(\log N)$</u>
  - depth can be as large as <u>$N - 1$</u>



Binary Tree Implementation

- since a binary tree has two children at most, we can keep <u>direct links</u> to each of them
  - element plus two pointers, left and right

```
struct BinaryNode
{
    Object      element;      // The data in the node
    BinaryNode *left;         // Left child
    BinaryNode *right;        // Right child
};
```

- drawn with circles and lines (graph)
- many applications, including compiler design

Tree Traversals

- easy to list all elements of a binary search tree in sorted order
  - inorder traversal
  - postorder traversal
  - preorder traversal
- implemented with recursive functions
- all $O(N)$

Tree Traversals

– inorder traversal

```
/**
 * Print the tree contents in sorted order.
 */
void printTree( ostream & out = cout ) const
{
    if( isEmpty( ) )
        out << "Empty tree" << endl;
    else
        printTree( root, out );
}

/**
 * Internal method to print a subtree rooted at t in sorted order.
 */
void printTree( BinaryNode *t, ostream & out ) const
{
    if( t != nullptr )
    {
        printTree( t->left, out );
        out << t->element << endl;
        printTree( t->right, out );
    }
}
```

Tree Traversals

- preorder traversal
  - visit node first, then left subtree, then right subtree
- postorder traversal
  - visit left subtree, right subtree, then node
- graphic technique for traversals
- level-order traversal
  - all nodes at depth *d* are processed before any node at depth *d* + *1*
    - not implemented with <u>recursion</u>
    - queue

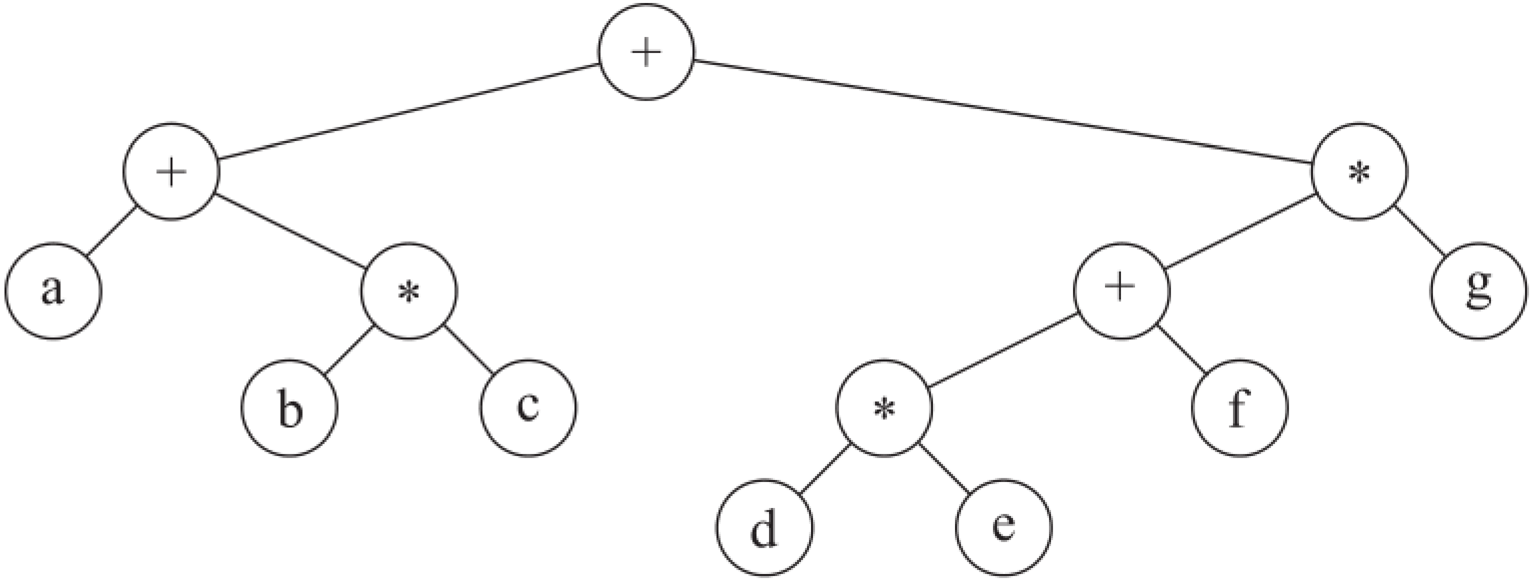
Tree Traversals

– height method using <u>postorder</u> traversal

```
/**
 * Internal method to compute the height of a subtree rooted at t.
 */
int height( BinaryNode *t )
{
    if( t == nullptr )
        return -1;
    else
        return 1 + max( height( t->left ), height( t->right ) );
}
```

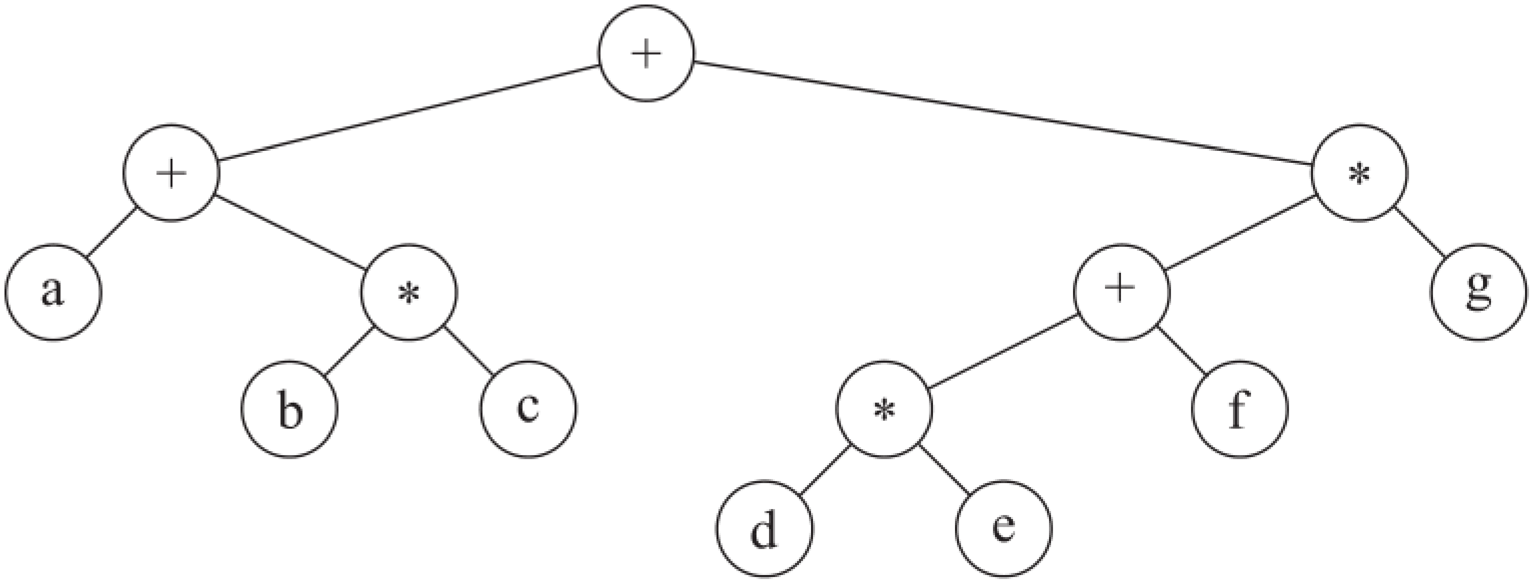
Binary Tree Example: Expression Trees

- expression tree
  - leaves represent <u>operands</u> (constants or variable names)
  - interior nodes represent <u>operators</u>
  - binary tree since most operators are binary, but not required
    - some operations are <u>unary</u>



Binary Tree Example: Expression Trees

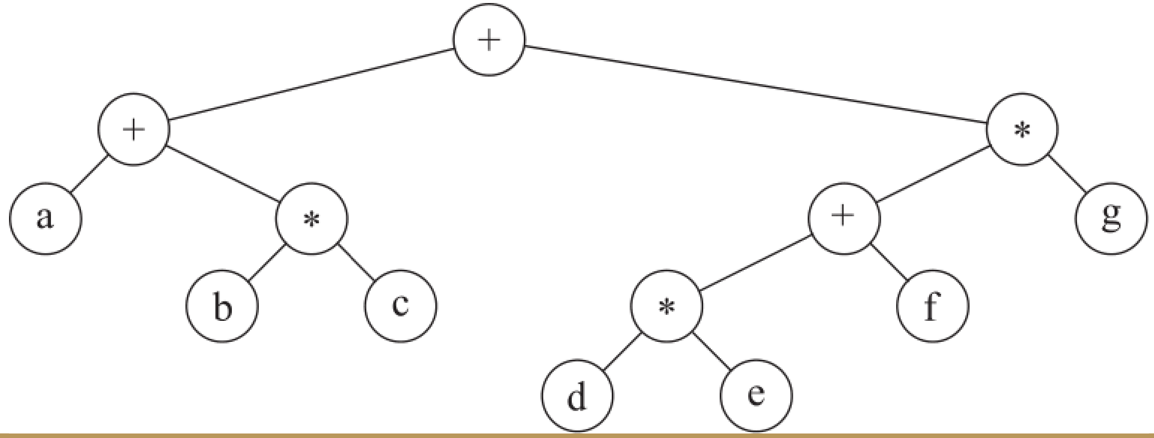
- evaluate expression tree, T, by applying operator at root to values obtained by <u>recursively</u> evaluating left and right subtrees



- left subtree: a + (b * c)
- right subtree: ((d * e) + f) * g
- <u>complete</u> tree: (a + (b * c)) + (((d * e) + f) * g)

Binary Tree Example: Expression Trees

- <u>inorder</u> traversal
  - recursively produce left expression
  - print operator at root
  - recursively produce right expression
- <u>postorder</u> traversal
  - result: a b c * + d e * f + g * +
- <u>preorder</u> traversal
  - result: + + a * b c * + * d e f g

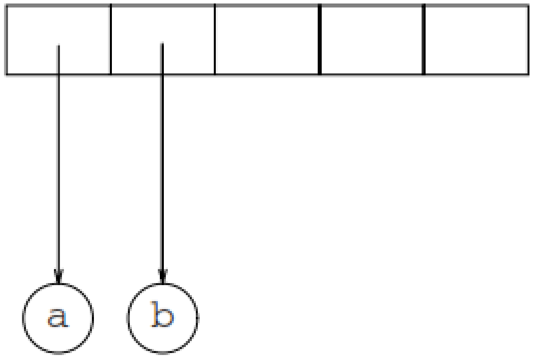


Binary Tree Example: Expression Trees

- goal: convert a postorder expression into an expression tree
  - read expression one symbol at a time
  - if operand, create node and push a pointer to it on the stack
  - if operator, pop pointers to two trees T_1 and T_2 from the stack
    - form new tree with operator as root
    - pointer to this tree is then pushed on the stack

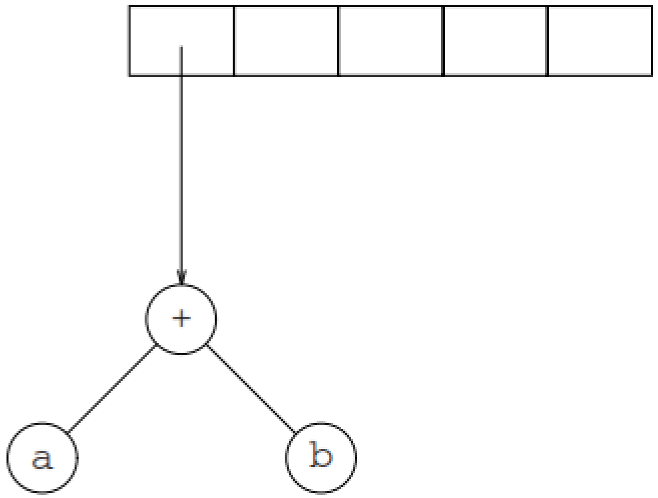
Binary Tree Example: Expression Trees

- example: a b + c d e + * *
- first two symbols are <u>operands</u> and are pushed on the stack



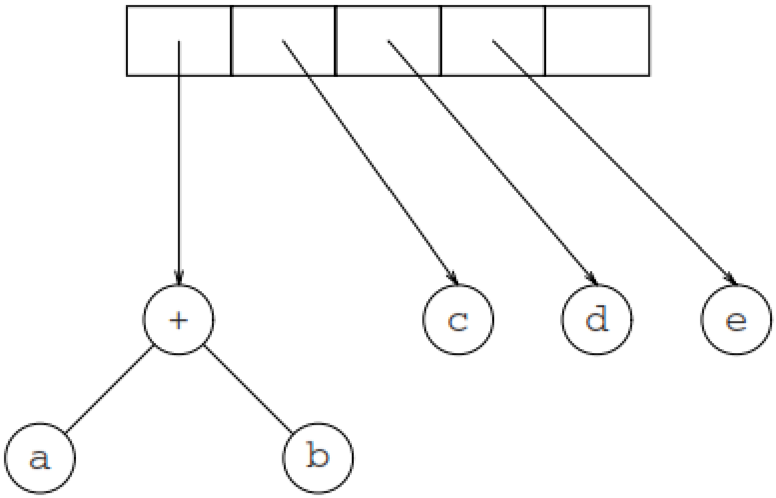
Binary Tree Example: Expression Trees

- example: a b + c d e + * *
- after + is read, two pointers are popped and new tree formed with a pointer pushed on the stack



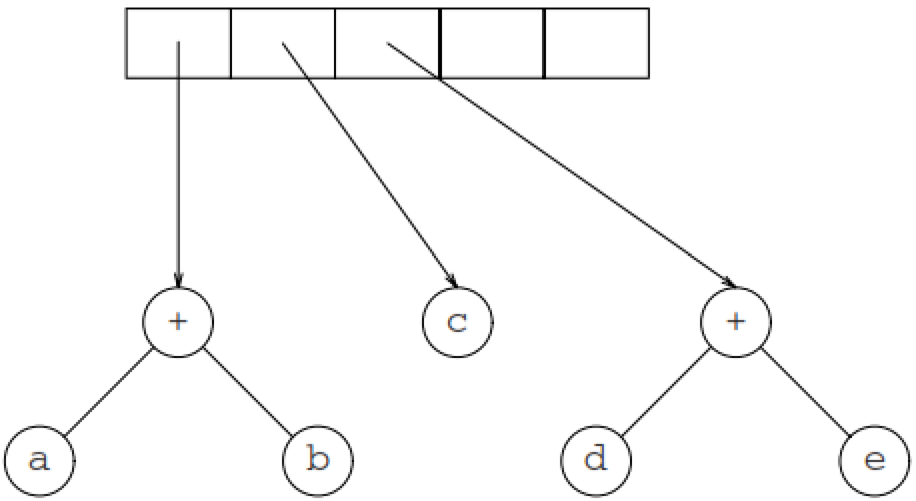
Binary Tree Example: Expression Trees

- example: a b + c d e + * *
- next, c, d, and e are read, with <u>one-node</u> tree created for each and pushed on the stack



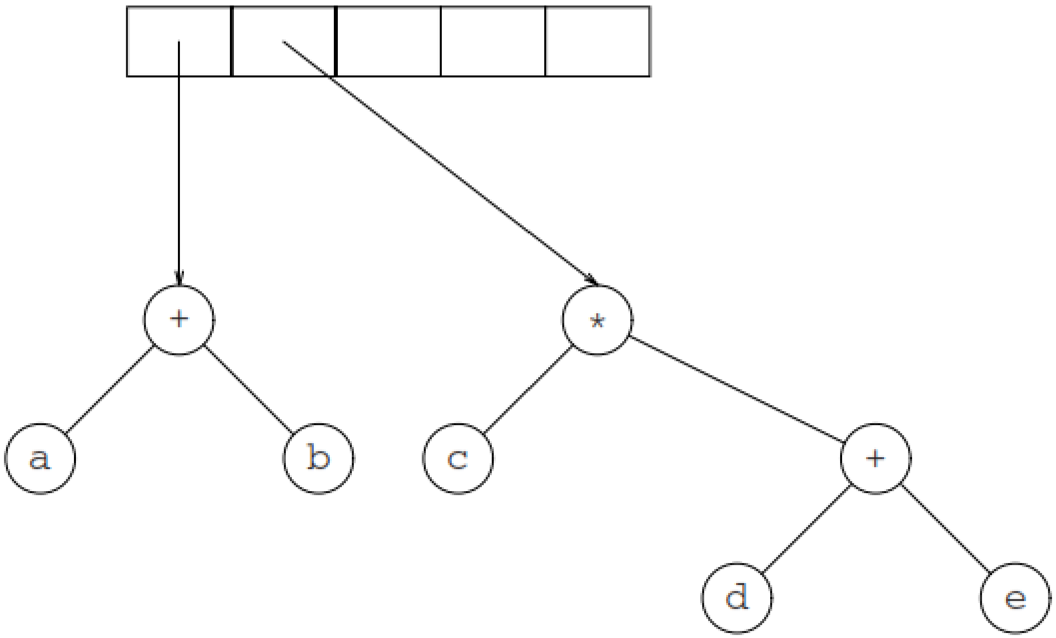
Binary Tree Example: Expression Trees

- example: a b + c d e + * *
- after + is read, two trees are merged



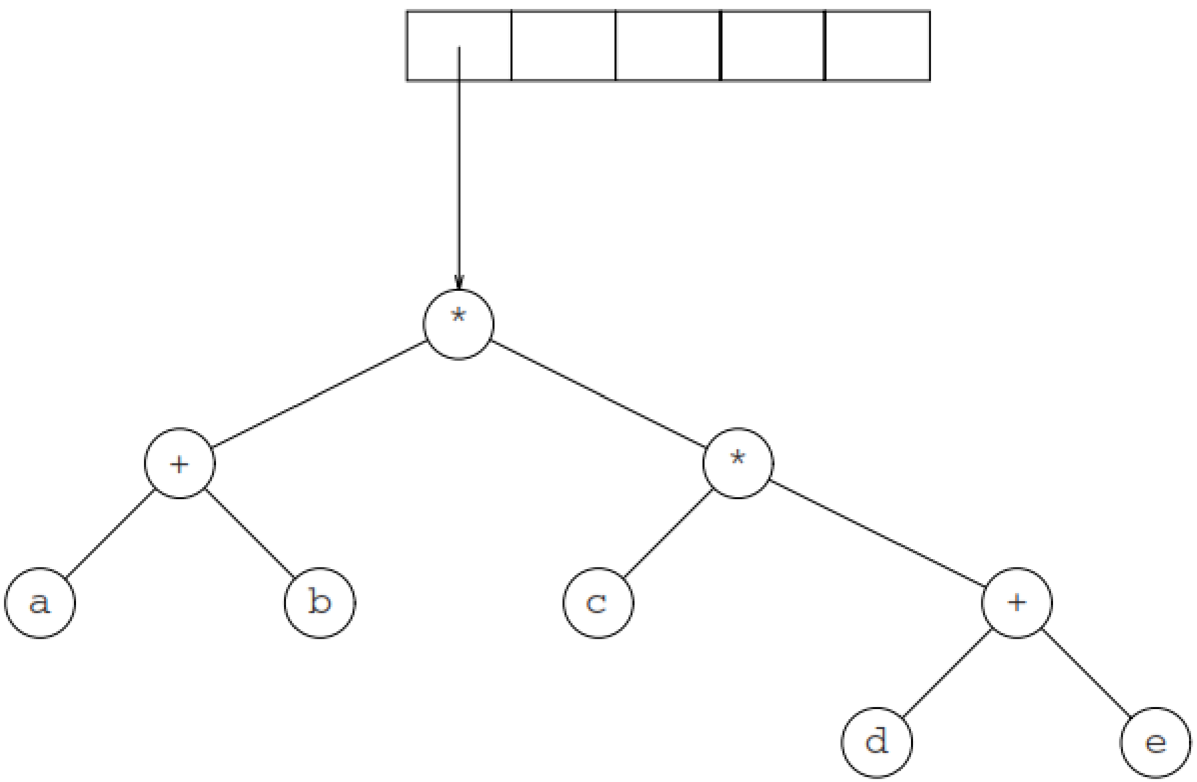
Binary Tree Example: Expression Trees

- example: a b + c d e + * *
- after * is read, two trees are popped to form a new tree with a * as root



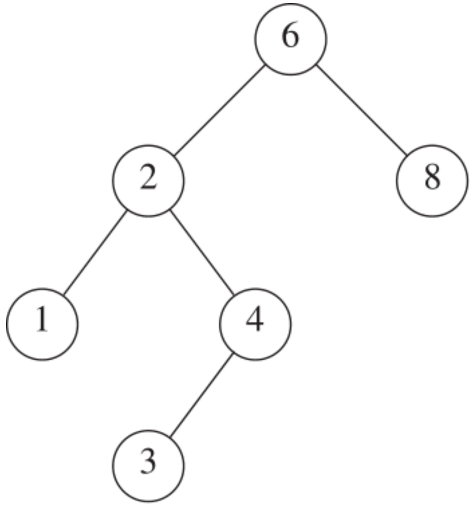
Binary Tree Example: Expression Trees

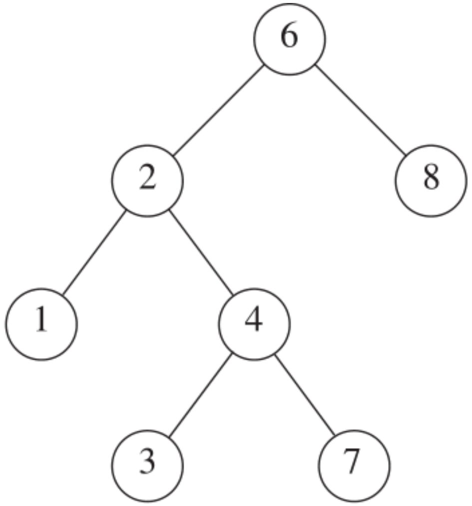
- example: a b + c d e + * *
- finally * is read, two trees are popped to form a <u>final</u> tree, which is left on the stack



Binary Search Tree ADT

- binary trees often used for <u>searching</u>
- assume each node in the tree stores one <u>element</u> (integer)
- binary search tree
  - for every node X in the tree
    - all items in left subtree are <u>smaller</u> than X
    - all items in right subtree are greater than X
  - items in tree must be order-able





Binary Search Tree ADT

- common operations on binary search trees
  - often written recursively
  - since average depth is $O(\log N)$, no worry about stack space
- binary search tree interface
  - searching depends on < operator, which must be defined for Comparable type
  - only data member is root pointer

Binary Search Tree ADT

```
template <typename Comparable>
class BinarySearchTree
{
  public:
    BinarySearchTree( );
    BinarySearchTree( const BinarySearchTree & rhs );
    BinarySearchTree( BinarySearchTree && rhs );
    ~BinarySearchTree( );

    const Comparable & findMin( ) const;
    const Comparable & findMax( ) const;
    bool contains( const Comparable & x ) const;
    bool isEmpty( ) const;
    void printTree( ostream & out = cout ) const;

    void makeEmpty( );
    void insert( const Comparable & x );
    void insert( Comparable && x );
    void remove( const Comparable & x );

    BinarySearchTree & operator=( const BinarySearchTree & rhs );
    BinarySearchTree & operator=( BinarySearchTree && rhs );
```

Binary Search Tree ADT

```
  private:
    struct BinaryNode
    {
        Comparable element;
        BinaryNode *left;
        BinaryNode *right;

        BinaryNode( const Comparable & theElement, BinaryNode *lt, BinaryNode *rt )
          : element{ theElement }, left{ lt }, right{ rt } { }

        BinaryNode( Comparable && theElement, BinaryNode *lt, BinaryNode *rt )
          : element{ std::move( theElement ) }, left{ lt }, right{ rt } { }
    };

    BinaryNode *root;

    void insert( const Comparable & x, BinaryNode * & t );
    void insert( Comparable && x, BinaryNode * & t );
    void remove( const Comparable & x, BinaryNode * & t );
    BinaryNode * findMin( BinaryNode *t ) const;
    BinaryNode * findMax( BinaryNode *t ) const;
    bool contains( const Comparable & x, BinaryNode *t ) const;
    void makeEmpty( BinaryNode * & t );
    void printTree( BinaryNode *t, ostream & out ) const;
    BinaryNode * clone( BinaryNode *t ) const;
};
```

Binary Search Tree ADT

– test for item in subtree

```
/**
 * Internal method to test if an item is in a subtree.
 * x is item to search for.
 * t is the node that roots the subtree.
 */
bool contains( const Comparable & x, BinaryNode *t ) const
{
    if( t == nullptr )
        return false;
    else if( x < t->element )
        return contains( x, t->left );
    else if( t->element < x )
        return contains( x, t->right );
    else
        return true;    // Match
}
```

Binary Search Tree ADT

- **`findMin`** and **`findMax`**
  - private methods return pointer to smallest/largest elements in the tree
  - to find the minimum, start at the root and go left as long as possible
  - similar for finding the maximum

Binary Search Tree ADT

– <u>recursive</u> version of **`findMin`**

```
/**
 * Internal method to find the smallest item in a subtree t.
 * Return node containing the smallest item.
 */
BinaryNode * findMin( BinaryNode *t ) const
{
    if( t == nullptr )
        return nullptr;
    if( t->left == nullptr )
        return t;
    return findMin( t->left );
}
```

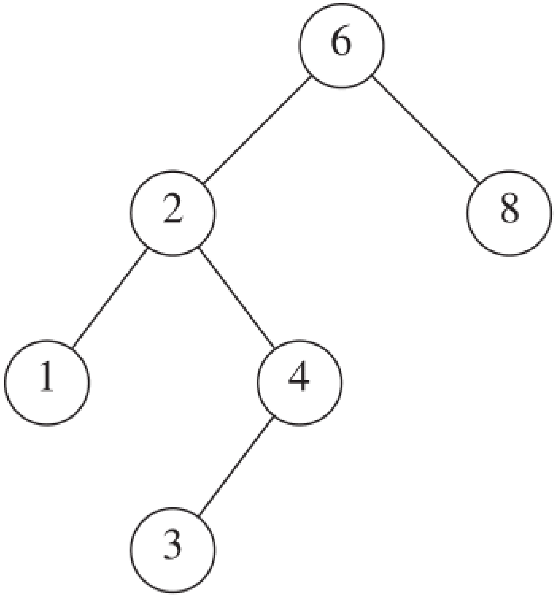
Binary Search Tree ADT

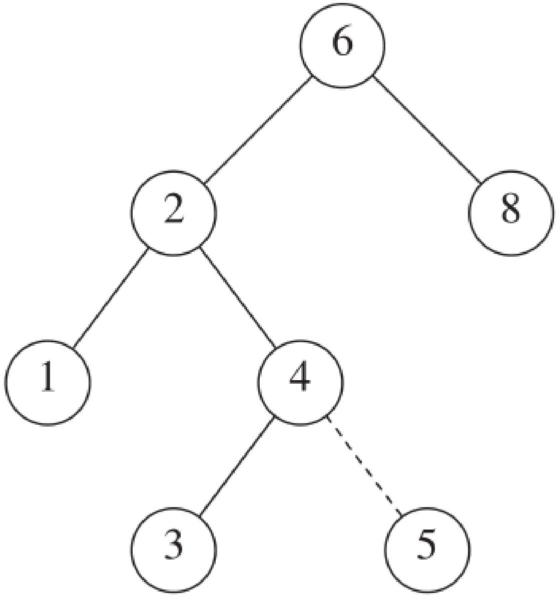
– nonrecursive version of **`findMax`**

```
/**
 * Internal method to find the largest item in a subtree t.
 * Return node containing the largest item.
 */
BinaryNode * findMax( BinaryNode *t ) const
{
    if( t != nullptr )
        while( t->right != nullptr )
            t = t->right;
    return t;
}
```

Binary Search Tree ADT

- insertion for binary search trees
  - to insert X into tree T, proceed <u>down</u> the tree, as in the **`contains`** function
  - if X is found, <u>do nothing</u>
  - otherwise, insert X at the last spot on the path traversed
  - example: insert 5 into binary search tree



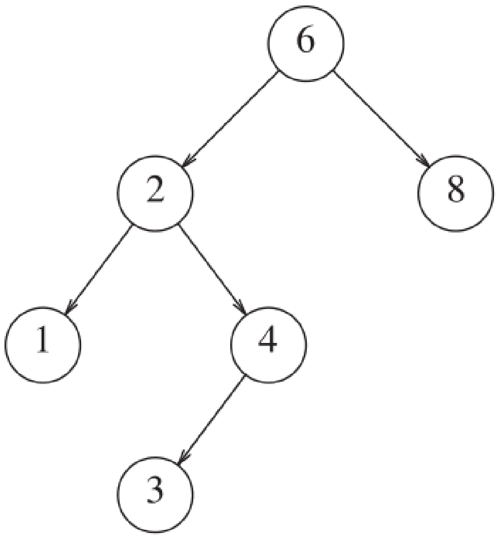


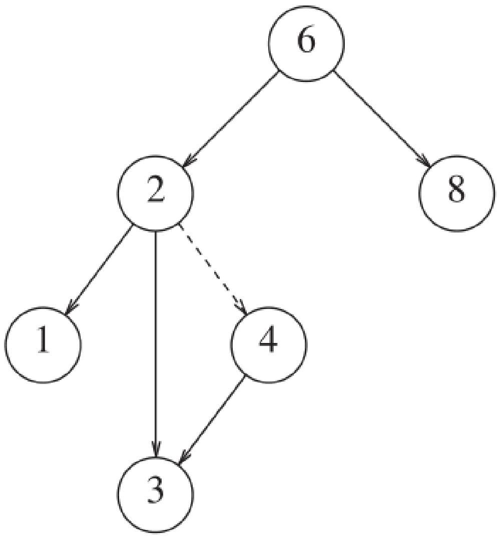
Binary Search Tree ADT

- duplicates can be handled by adding a <u>count</u> to the node record
  - better than inserting <u>duplicates</u> in tree
  - may not work well if key is only small part of larger structure

Binary Search Tree ADT

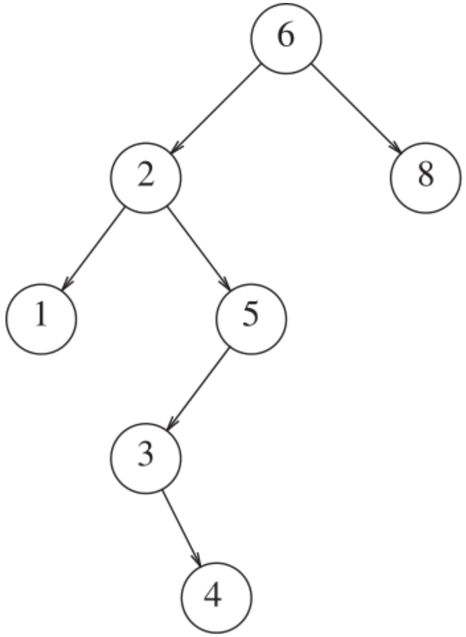
- deletion in binary search tree may be difficult
- multiple cases
  - if node is leaf, it can be deleted immediately
  - if node has only one child, node can be deleted after its parent adjusts a link to bypass the node

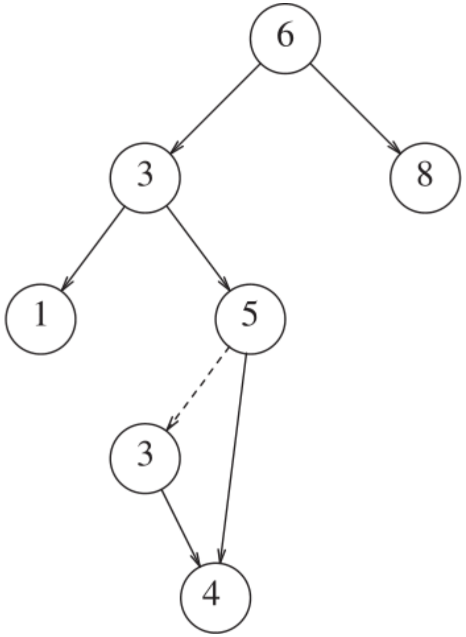




Binary Search Tree ADT

- multiple cases (cont.)
  - complicated case: node with <u>two</u> children
    - replace data of this node with smallest data of right subtree and recursively delete the node
    - since smallest node in right subtree cannot have a left child, the second remove is easy





Binary Search Tree ADT

- if number of deletions small, lazy deletion may be used
  - node is marked deleted rather than actually being deleted
  - especially popular if duplicates allowed
    - count of duplicates can be decremented
  - incurs only small penalty on tree since height not affected greatly
  - if deleted node reinstated, some benefits

Binary Search Tree Average-Case Analysis

- we expect most operations on binary search trees will have <u>$O(\log N)$</u> time
  - average depth over all nodes can be shown to be $O(\log N)$
  - all insertions and deletions must be equally <u>likely</u>
  - sum of the depths of all nodes in a tree is known as <u>internal path length</u>

Binary Search Tree Average-Case Analysis

- the run time of binary search trees depends on the <u>depth</u> of the tree, which in turn depends on the <u>order</u> that the keys are inserted
- let $D(N)$ be the internal path length for a tree of N nodes
- we know that $D(1) = 0$
- a tree of an i-node left subtree and an $(N - i - 1)$-node right subtree, plus a root at depth zero for $0 \leq i \leq N$
- total number of nodes in tree = left subtree + right subtree + 1
- all nodes except the <u>root</u> are one level deeper,

$$D(N) = D(i) + D(N - i - 1) + N - 1$$

Binary Search Tree Average-Case Analysis

- if all subtree sizes are <u>equally likely</u>, then the average for each subtree is

$$\left(\frac{1}{N}\right)\sum_{j=1}^{N-1} D(j)$$

therefore, for the total number of nodes

$$D(N) = \left(\frac{2}{N}\right)\left[\sum_{j=1}^{N-1} D(j)\right] + N - 1$$

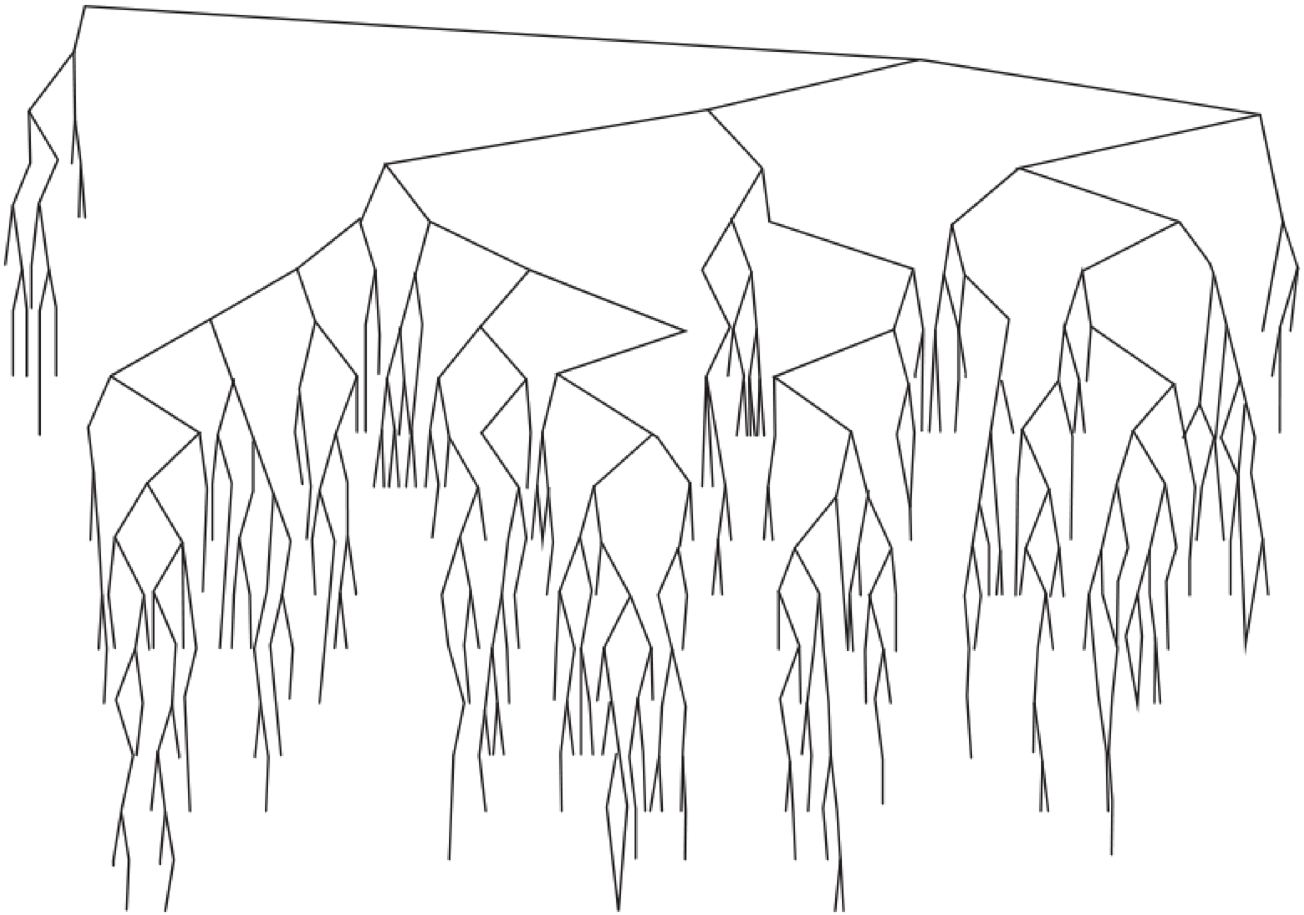
- once this recurrence relation is evaluated, the result is

$$D(N) = O(N \log N)$$

and the <u>average</u> number of nodes is $O(\log N)$

Binary Search Tree Average-Case Analysis

– example: randomly generated 500-node tree has <u>expected</u> depth of 9.98

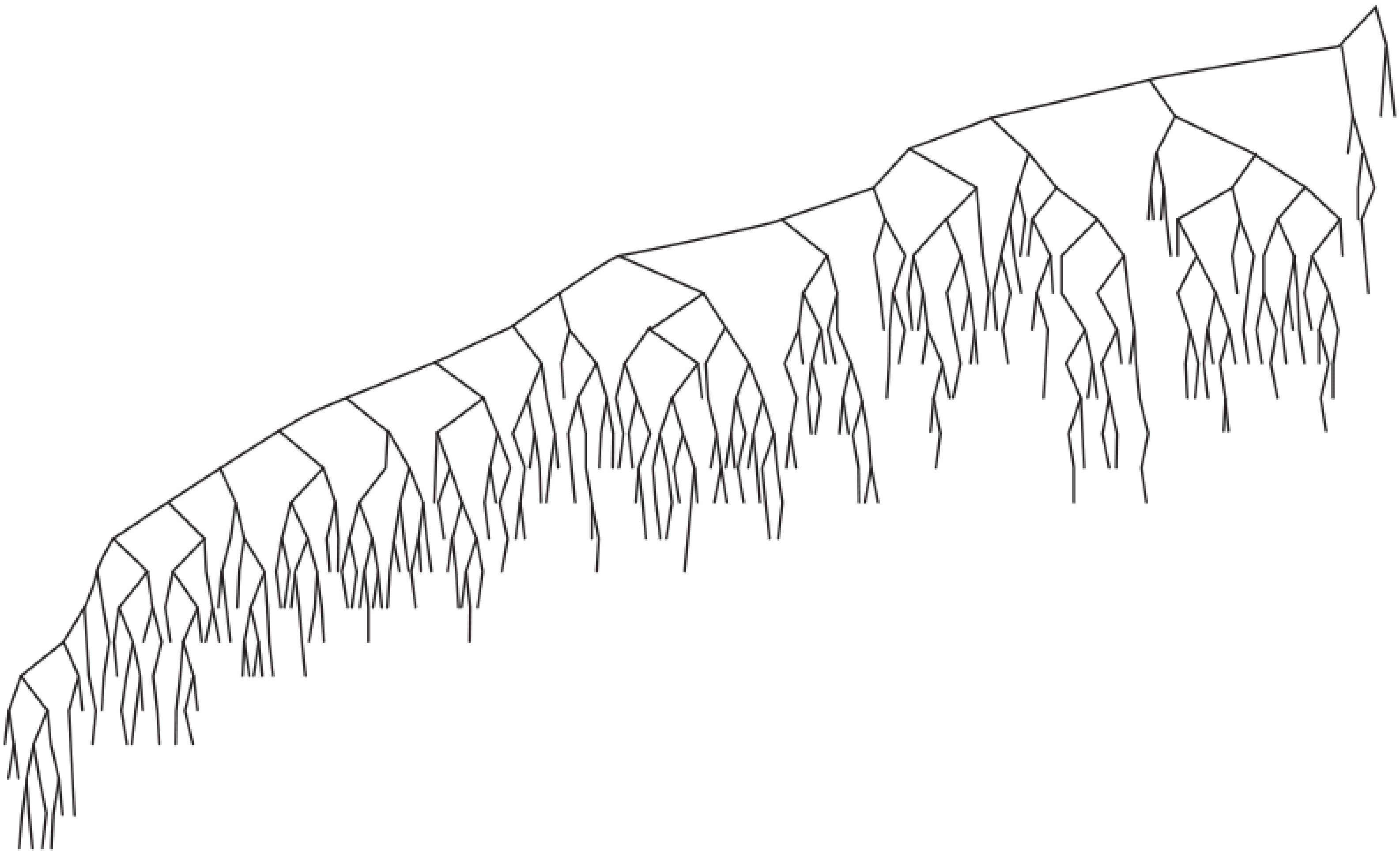


Binary Search Tree Average-Case Analysis

- deletions, however, bias the left subtrees to be longer because we always replace a deleted node with a node from the right subtree
- exact effect of deletions still unknown
- if insertions and deletions are alternated $\Theta(N^2)$ times, then expected depth is $\Theta\left(\sqrt{N}\right)$

Binary Search Tree Average-Case Analysis

- after 250,000 random insert/delete pairs, tree becomes unbalanced, with depth = 12.51

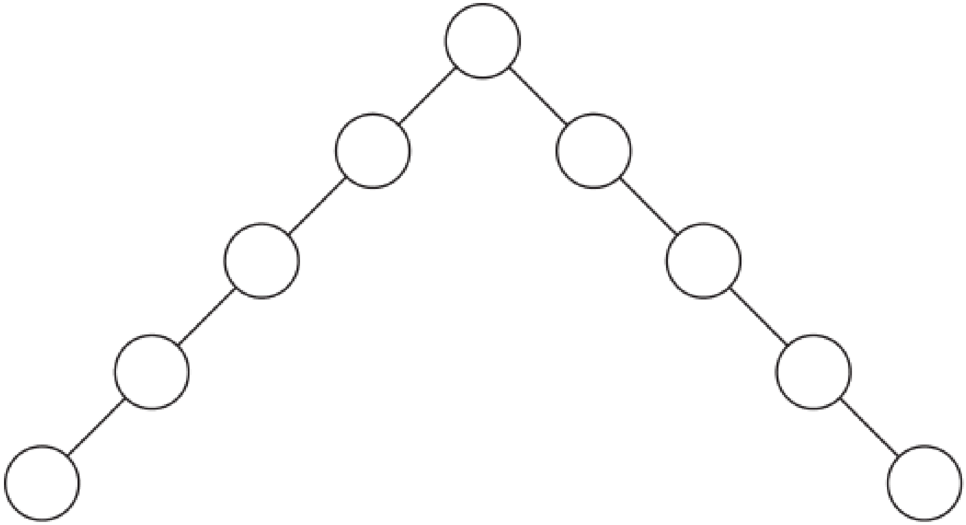


Binary Search Tree Average-Case Analysis

- could randomly choose between smallest element in the right subtree and largest element in the left subtree when replacing deleted element
  - should keep <u>bias</u> low, but not yet proven
- bias does not show up for <u>small</u> trees
- if $o(N^2)$ insert/remove pairs used, tree actually gains balance
- average case analysis extremely difficult
- two possible solutions
  - balanced trees
  - self-adjusting trees

AVL Trees

- Adelson-Velskii and Landis (AVL) tree is a binary search tree with a balance condition
- balance condition in general
  - must be easy to maintain
  - ensures depth of tree is $O(\log N)$
- simplest idea: left and right subtrees have the same height
  - does not always work

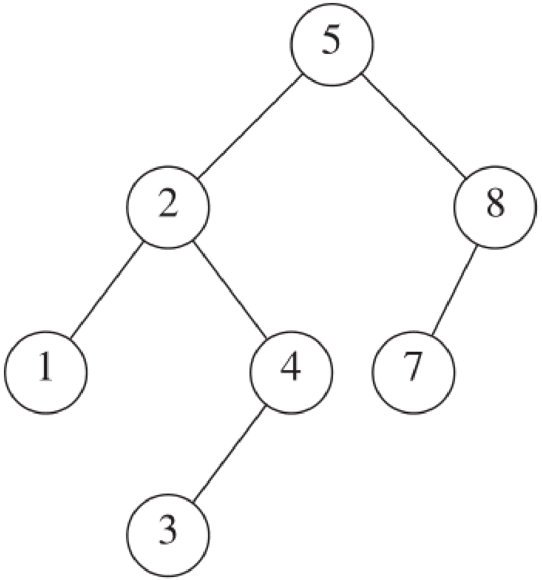


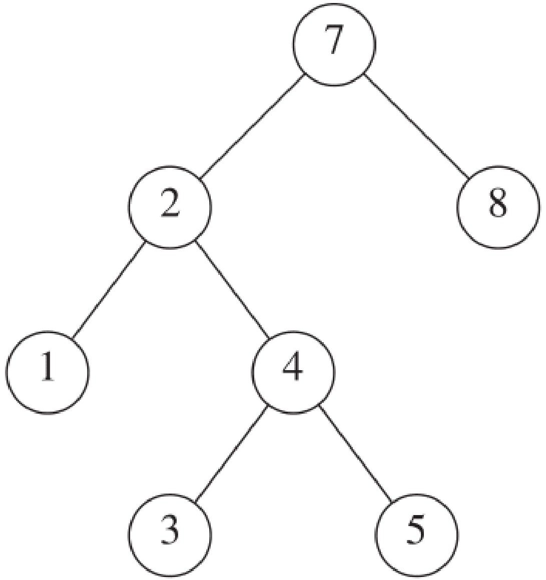
AVL Trees

- alternate balance condition: <u>every node</u> must have left and right subtrees of the same height
  - only <u>perfectly</u> balanced trees of $2^k - 1$ nodes would work
  - condition too rigid

AVL Trees

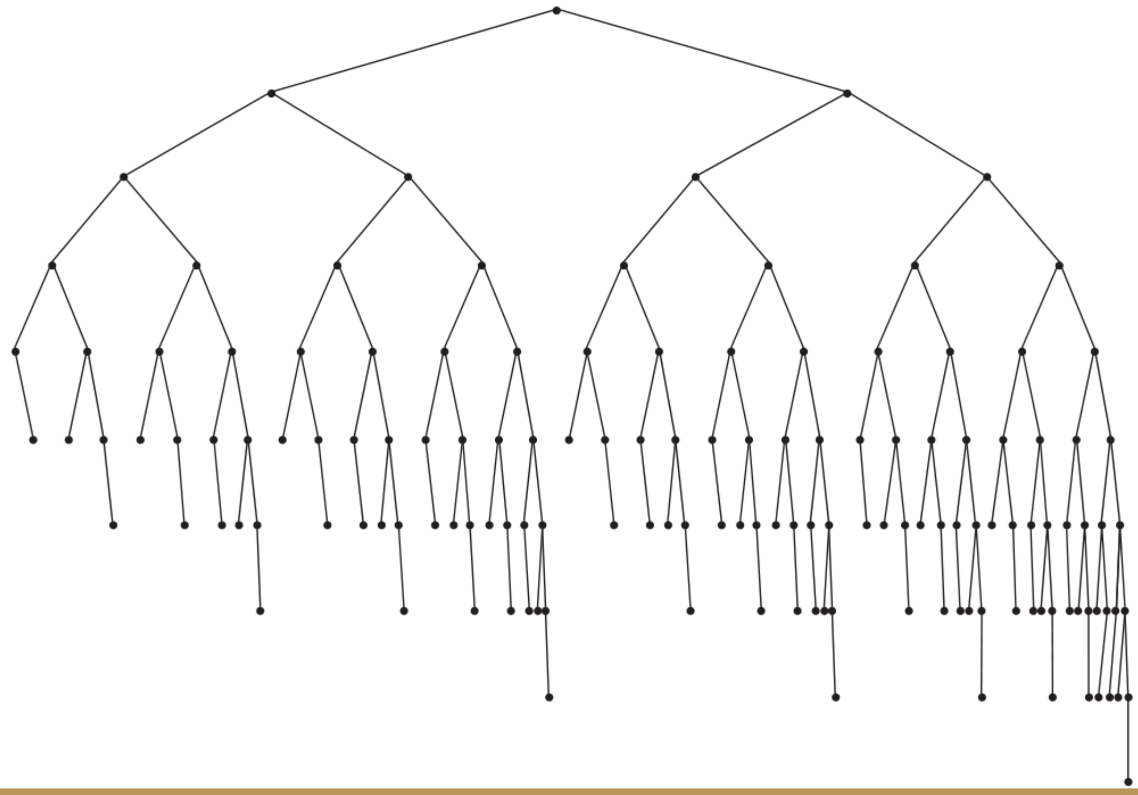
- AVL tree
  - for each node in the tree, height of left and right subtrees differ by at most <u>1</u>
    - height <u>balance</u> = height of right subtree – height of left
    - height of an empty tree: -1
    - height information kept in the node structure





AVL Trees

- example AVL tree
  - fewest nodes for a tree of height 9
  - left subtree contains fewest nodes for height 7
  - right subtree contains fewest nodes for height 8



AVL Trees

- <u>minimum</u> number of nodes, $S(h)$, in an AVL tree of height h

$$S(h) = S(h-1) + S(h-2) + 1 \qquad S(0) = 1, \qquad S(1) = 2$$

  - closely related to <u>Fibonacci</u> numbers
- all operations can be performed in $O(\log N)$ time, except <u>insertion and deletion</u>

AVL Trees

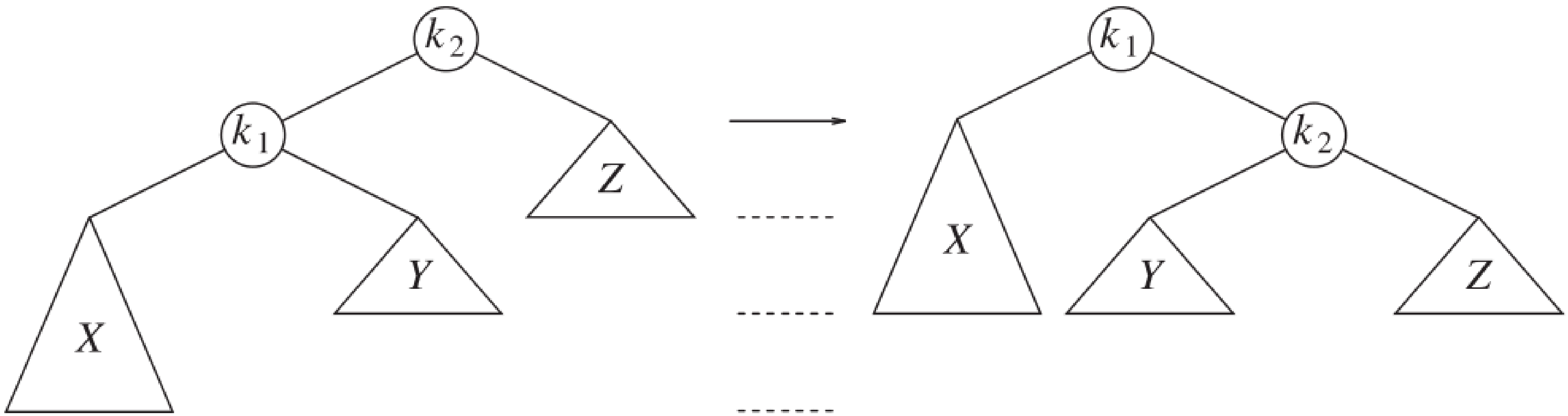
- insertion
  - update all balance information in the nodes on the path back to the root
  - could violate the balance condition
  - rotations used to restore the balance property
- deletion
  - perform same promotion as in a binary search tree, updating the balance information as necessary
    - same balancing operations for insertion can then be used

AVL Trees

- if α is the node requiring <u>rebalancing</u> (the heights of its left and right subtrees differ by 2), the violation occurred in one of four cases
  - an insertion into the left subtree of the left child of α
  - an insertion into the right subtree of the left child of α
  - an insertion into the left subtree of the right child of α
  - an insertion into the right subtree of the right child of α
- cases 1 and 4 are mirror image symmetries with respect to α and can be resolved with a <u>single</u> rotation
- cases 2 and 3 are mirror image symmetries with respect to α and can be resolved with a <u>double</u> rotation

AVL Trees

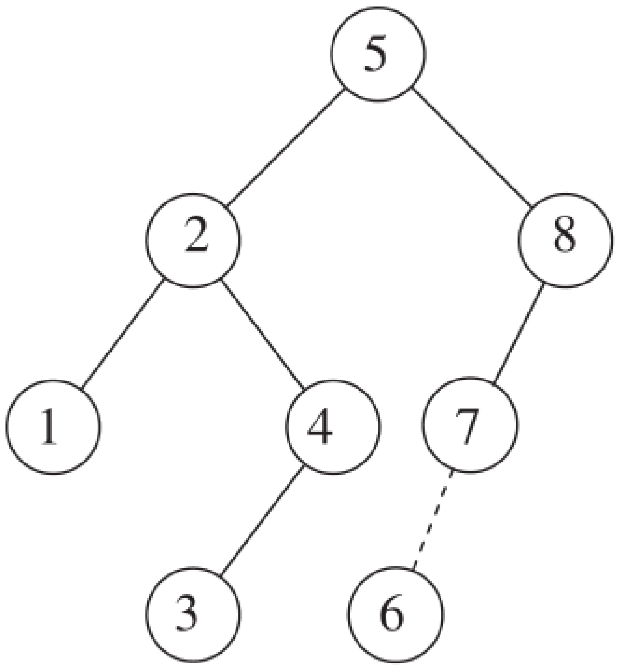
- single rotation
  - only possible case 1 scenario
  - to balance, imagine "picking up" tree by k_1

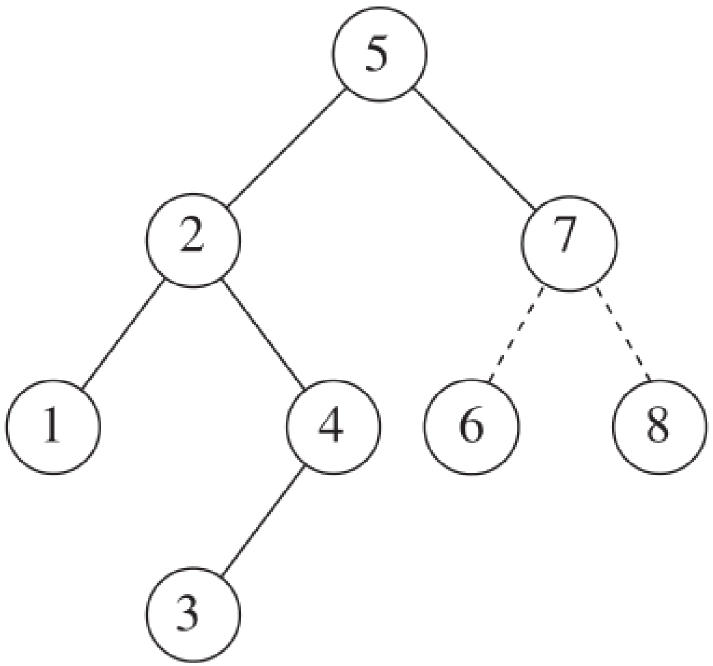


  - new tree has <u>same height</u> as original tree

AVL Trees

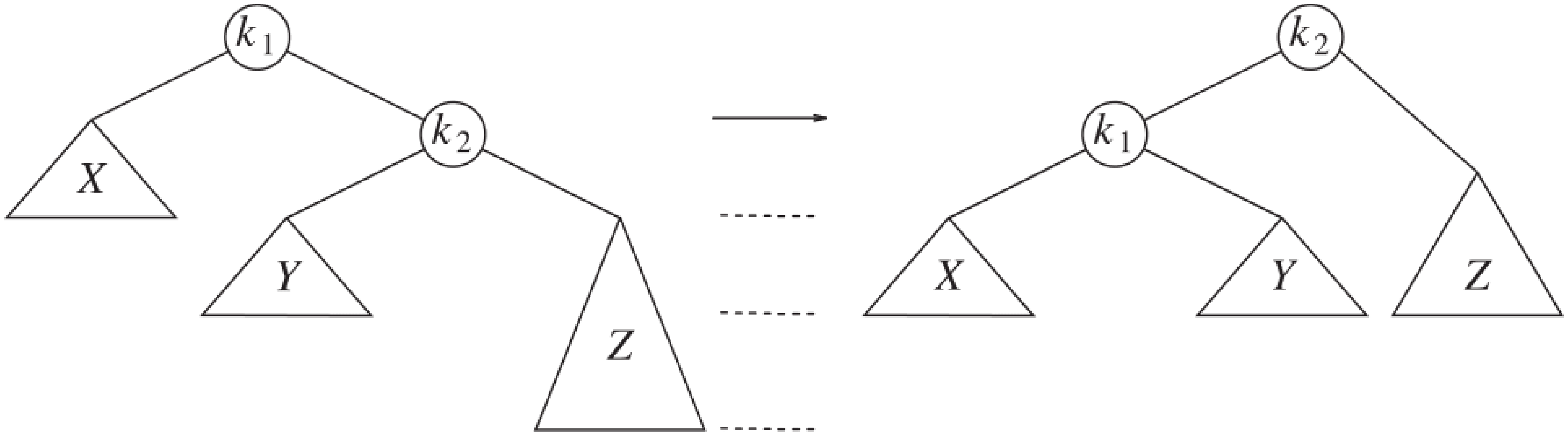
- example tree
  - when adding 6, node 8 becomes <u>unbalanced</u>
  - to balance, perform single rotation between 7 and 8





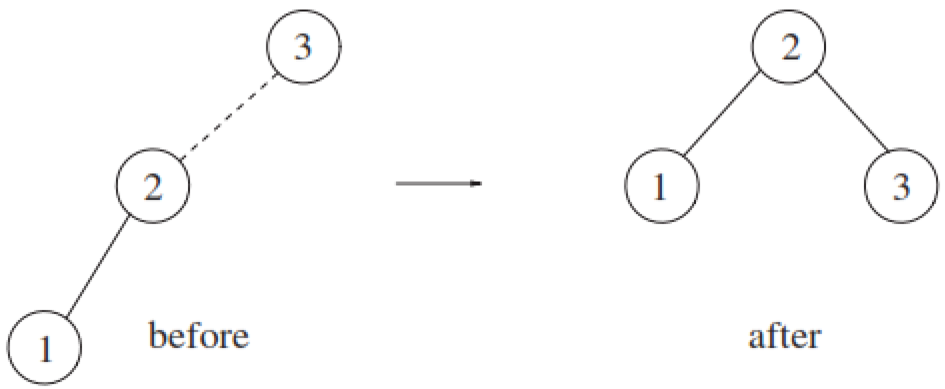
AVL Trees

- example tree
  - <u>symmetric</u> case for case 4



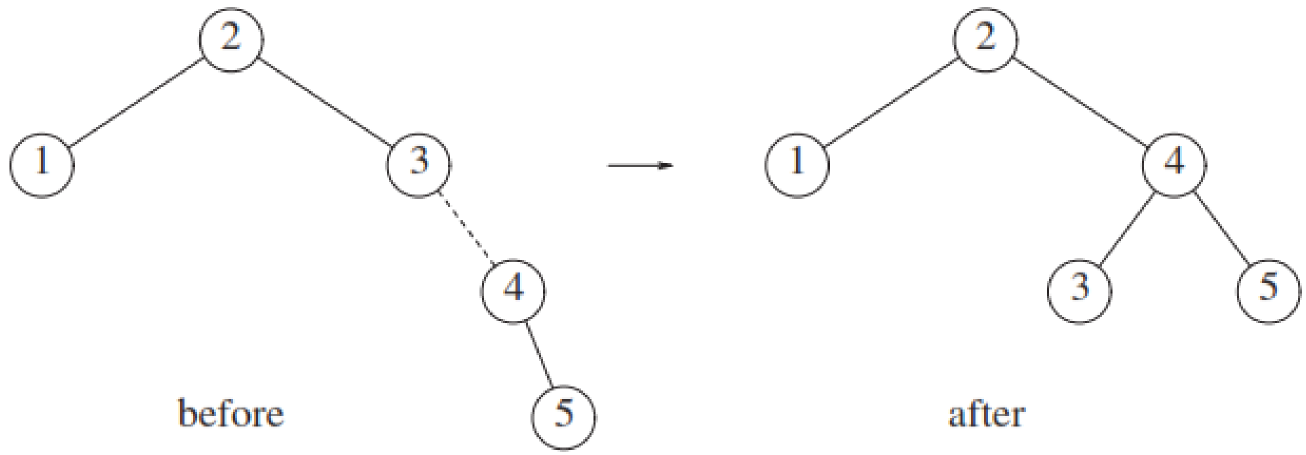
AVL Trees

- example
  - insert 3, 2, and 1 into an empty tree



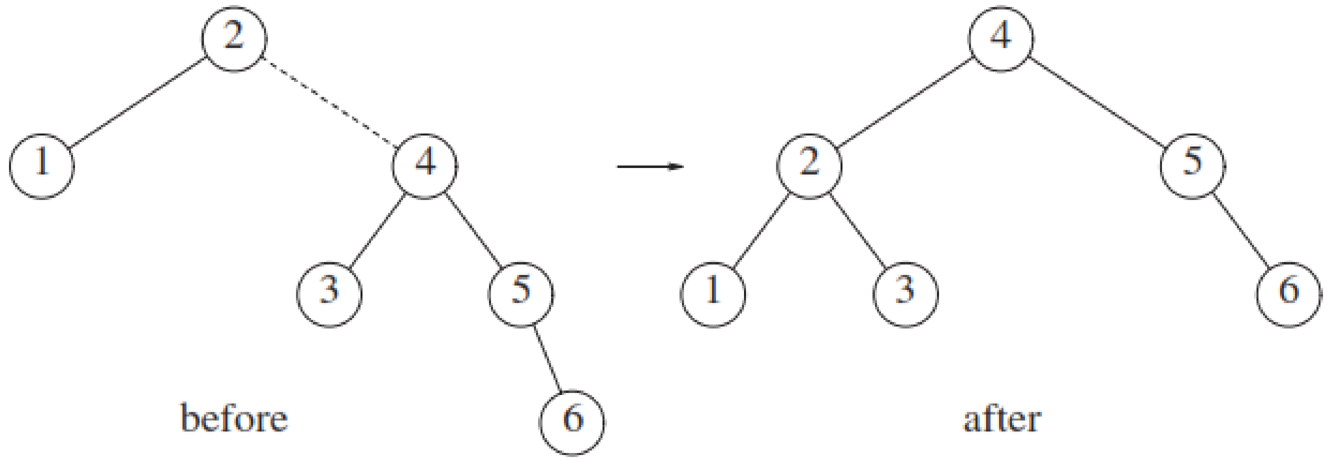
AVL Trees

- example
  - insert 4 and 5



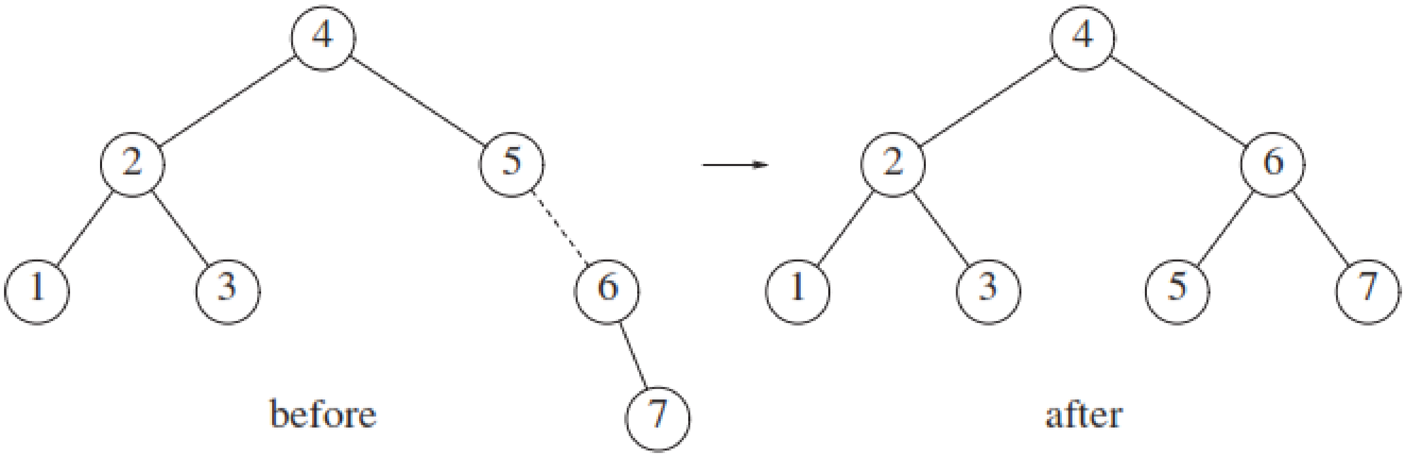
AVL Trees

- example
  - insert 6



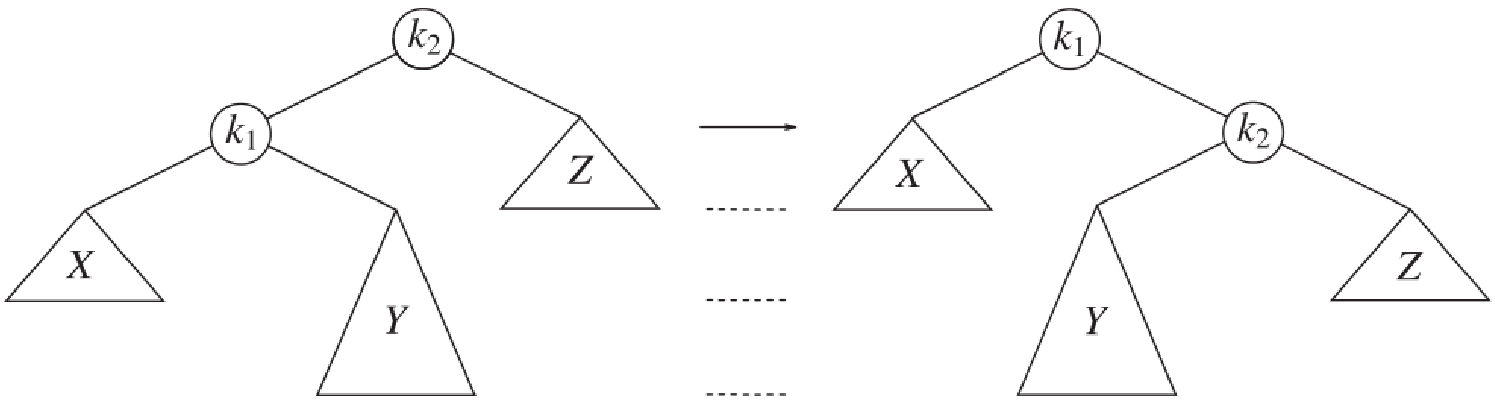
AVL Trees

- example
  - insert 7



AVL Trees

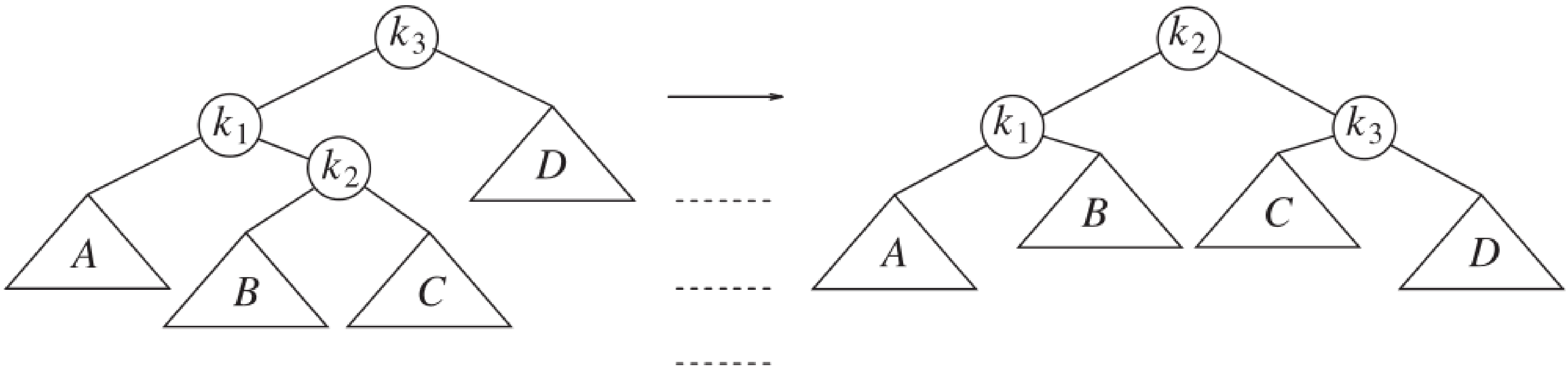
- double rotation
  - for cases 2 and 3, a <u>single</u> rotation will not work



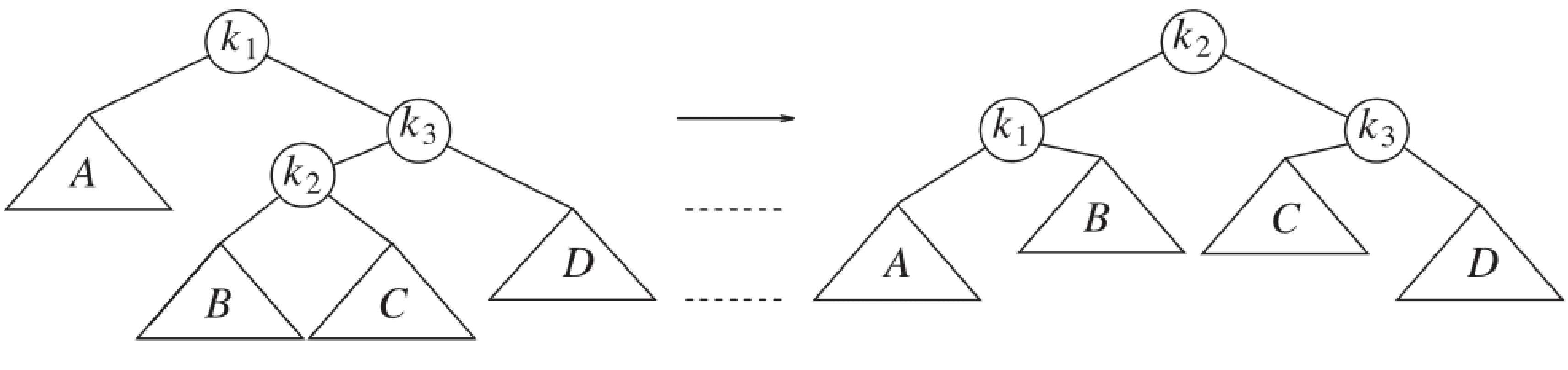
  - tree Y can be expanded to a node with two <u>subtrees</u>

AVL Trees

– double rotation
  – left-right double rotation for case 2

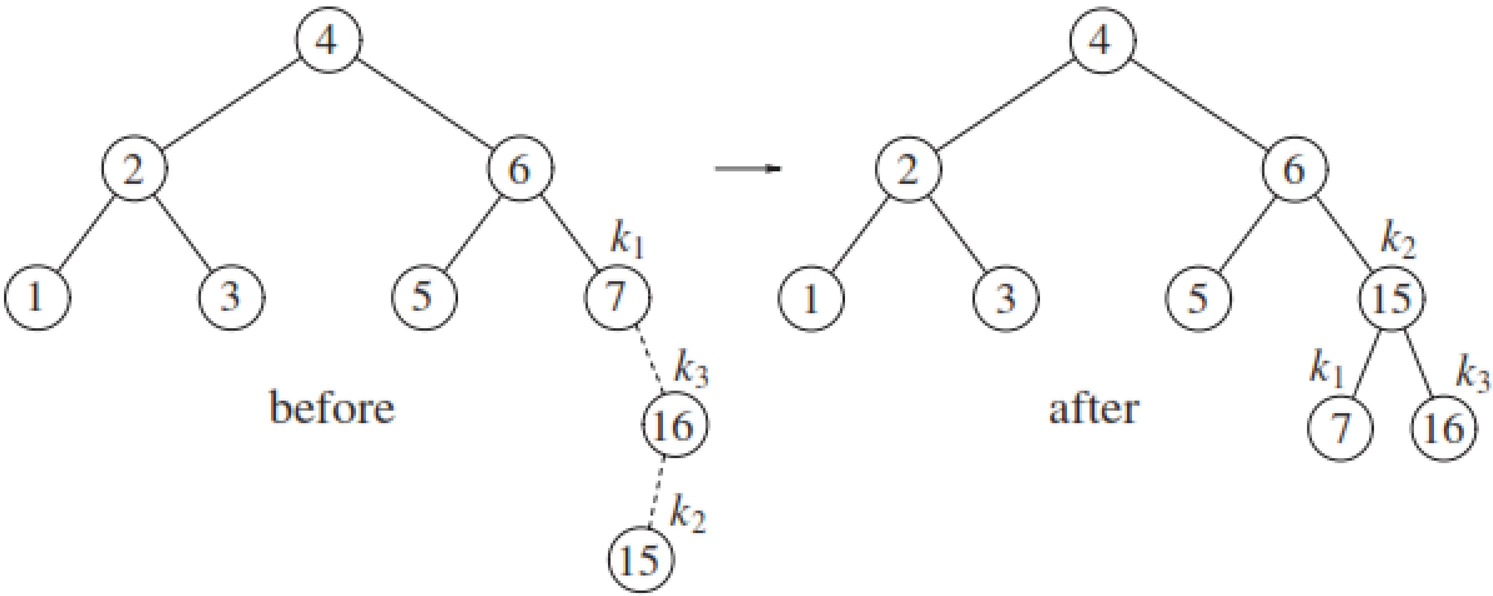


  – right-left double rotation for case 3



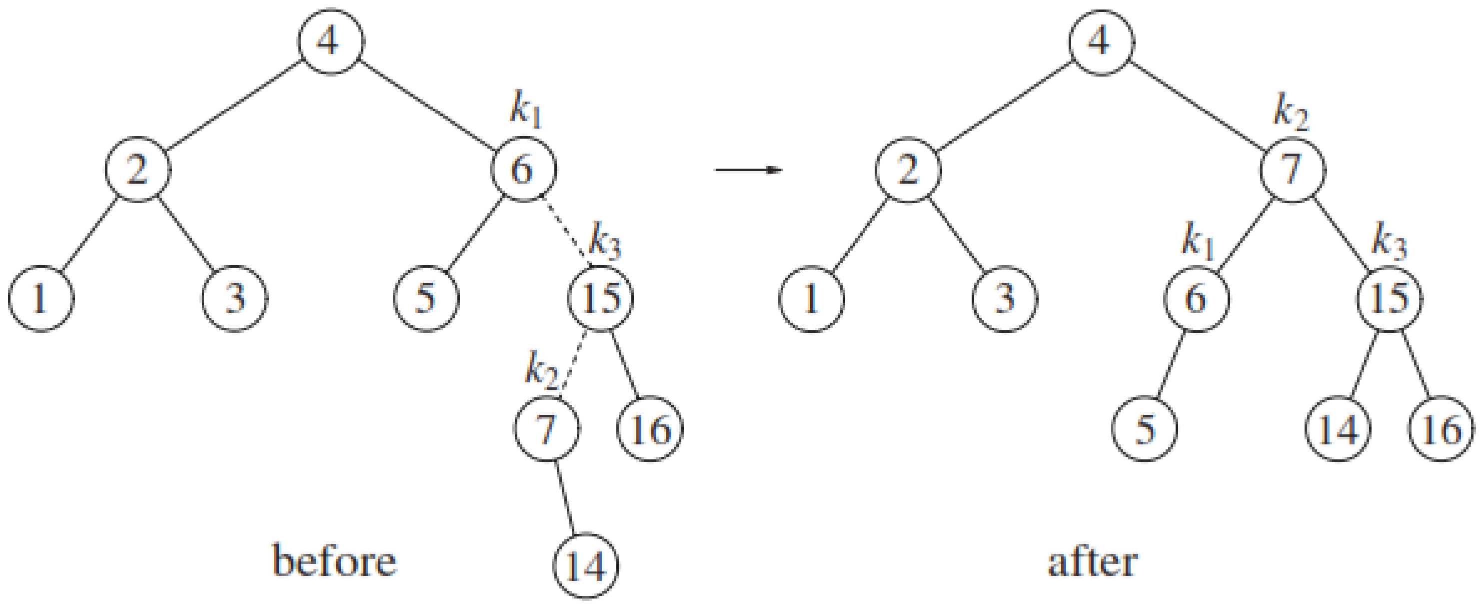
AVL Trees

- example
  - insert 16 and 15



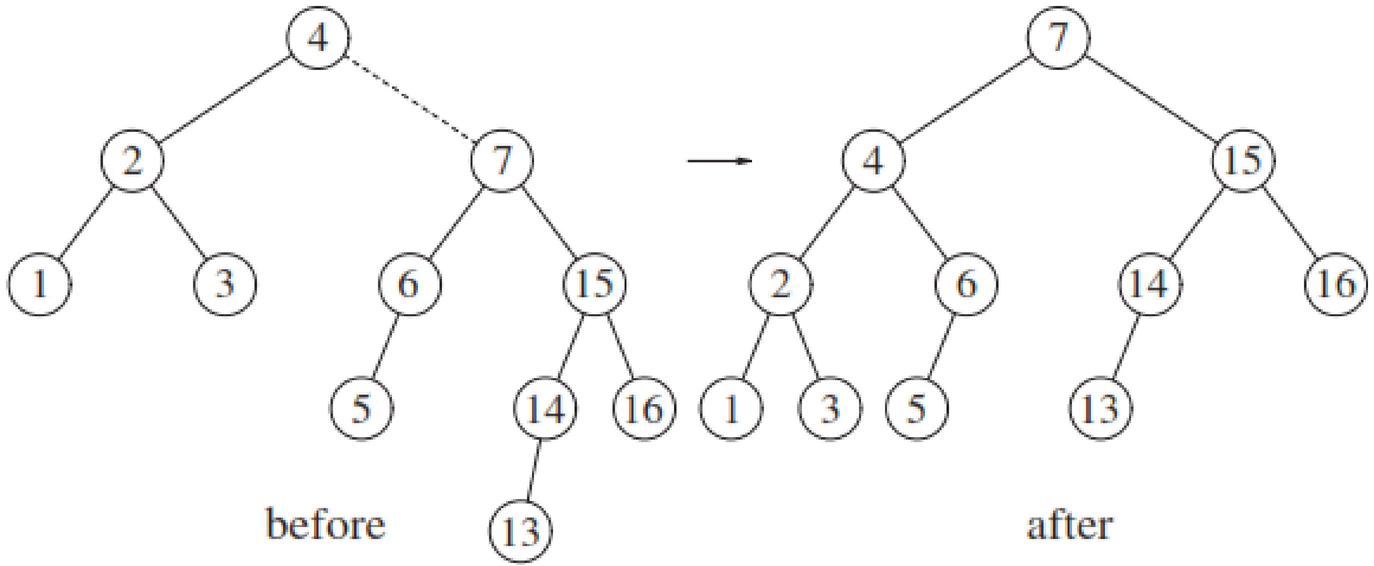
AVL Trees

- example
  - insert 14



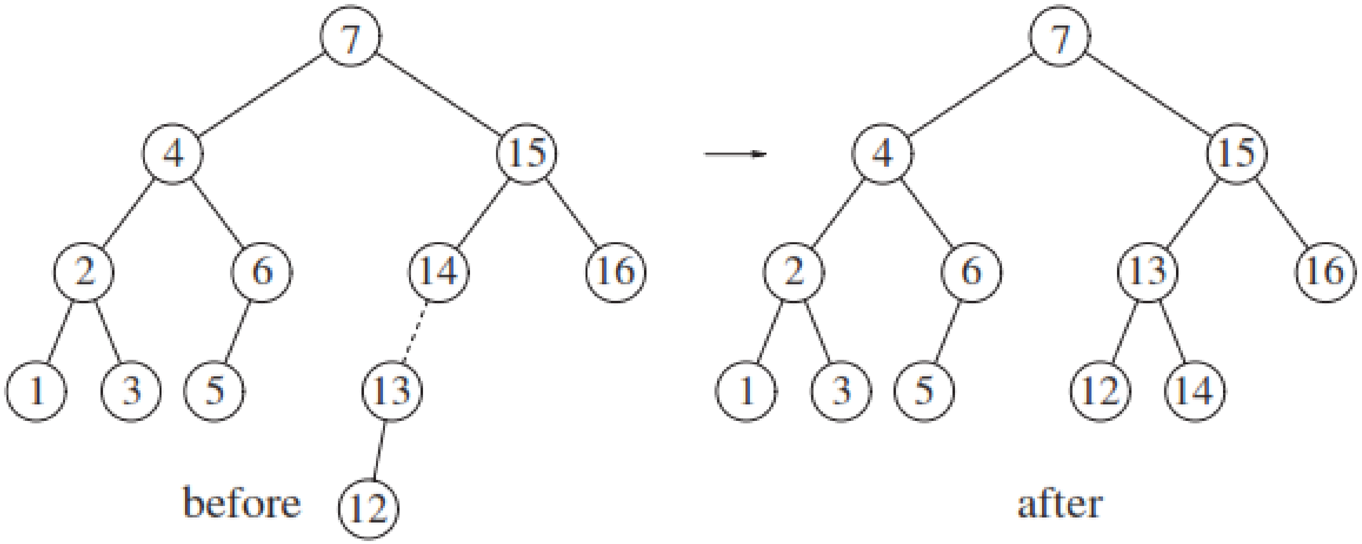
AVL Trees

- example
  - insert 13



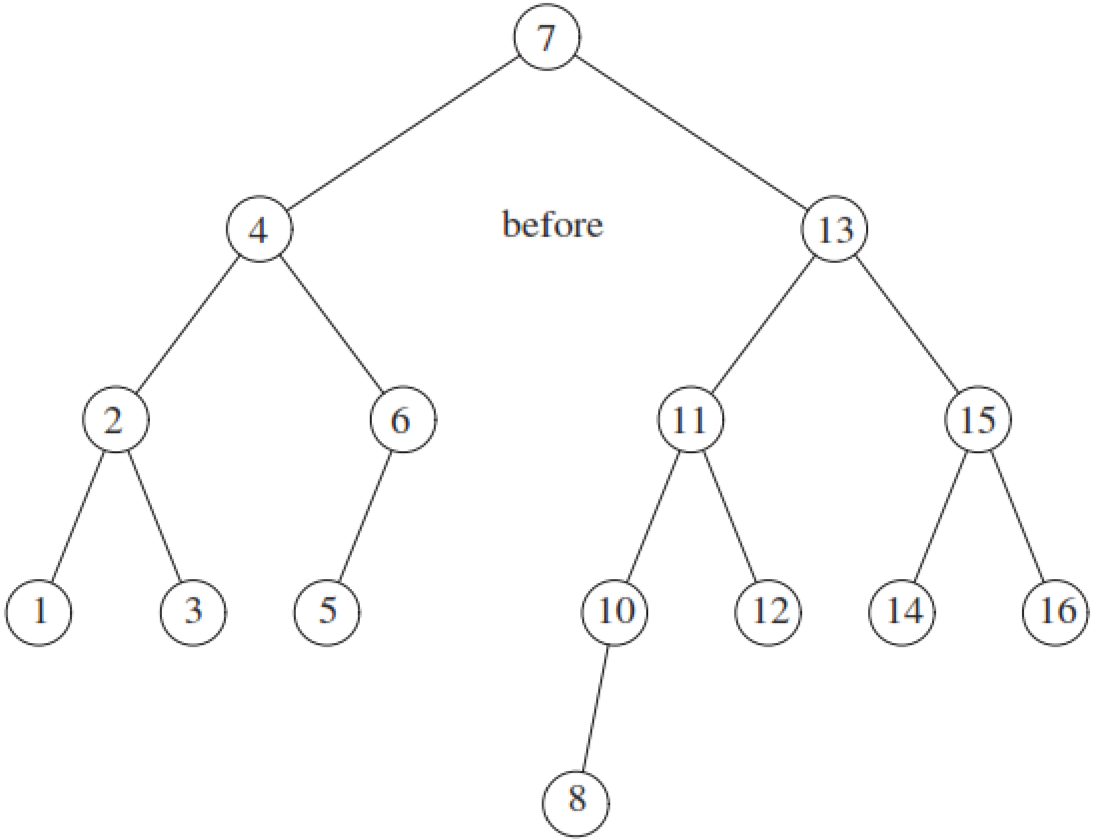
AVL Trees

- example
  - insert 12



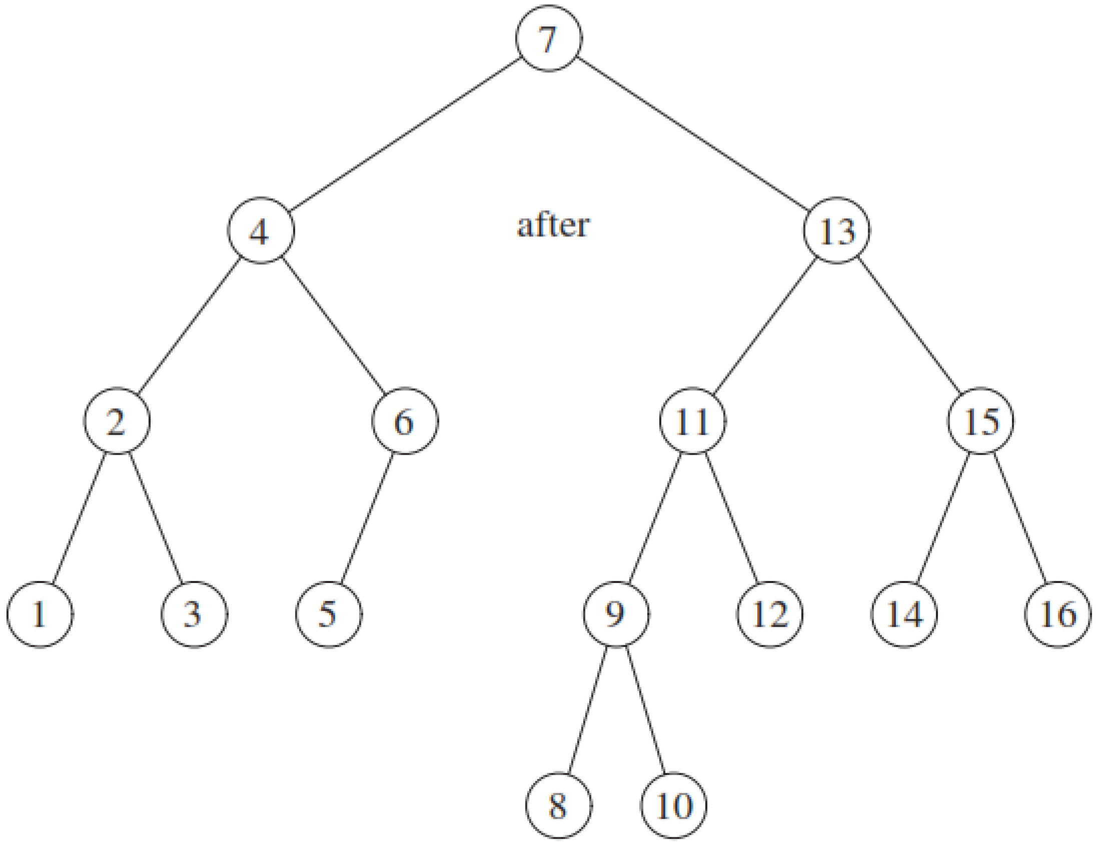
AVL Trees

- example
  - insert 11, 10, and 8



AVL Trees

- example
  - insert 9



AVL Trees

- implementation
  - node definition

```
struct AvlNode
{
    Comparable element;
    AvlNode   *left;
    AvlNode   *right;
    int       height;

    AvlNode( const Comparable & ele, AvlNode *lt, AvlNode *rt, int h = 0 )
      : element{ ele }, left{ lt }, right{ rt }, height{ h } { }

    AvlNode( Comparable && ele, AvlNode *lt, AvlNode *rt, int h = 0 )
      : element{ std::move( ele ) }, left{ lt }, right{ rt }, height{ h } { }
};
```

AVL Trees

- implementation
  - function to compute height of AVL node

```
/**
 * Return the height of node t or -1 if nullptr.
 */
int height( AvlNode *t ) const
{
    return t == nullptr ? -1 : t->height;
}
```

AVL Trees

- implementation
  - insertion

```
/**
 * Internal method to insert into a subtree.
 * x is the item to insert.
 * t is the node that roots the subtree.
 * Set the new root of the subtree.
 */
void insert( const Comparable & x, AvlNode * & t )
{
    if( t == nullptr )
        t = new AvlNode{ x, nullptr, nullptr };
    else if( x < t->element )
        insert( x, t->left );
    else if( t->element < x )
        insert( x, t->right );

    balance( t );
}
```

AVL Trees

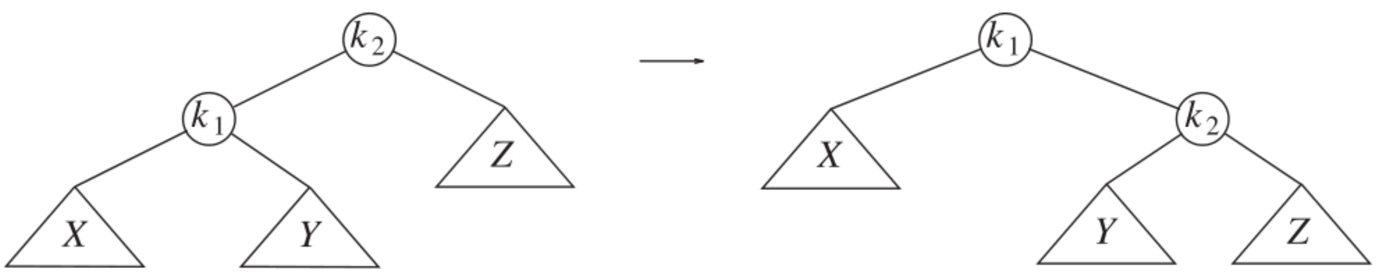
– implementation

```
static const int ALLOWED_IMBALANCE = 1;

// Assume t is balanced or within one of being balanced
void balance( AvlNode * & t )
{
    if( t == nullptr )
        return;

    if( height( t->left ) - height( t->right ) > ALLOWED_IMBALANCE )
        if( height( t->left->left ) >= height( t->left->right ) )
            rotateWithLeftChild( t );
        else
            doubleWithLeftChild( t );
    else
    if( height( t->right ) - height( t->left ) > ALLOWED_IMBALANCE )
        if( height( t->right->right ) >= height( t->right->left ) )
            rotateWithRightChild( t );
        else
            doubleWithRightChild( t );

    t->height = max( height( t->left ), height( t->right ) ) + 1;
}
```

AVL Trees

- implementation
  - single rotation

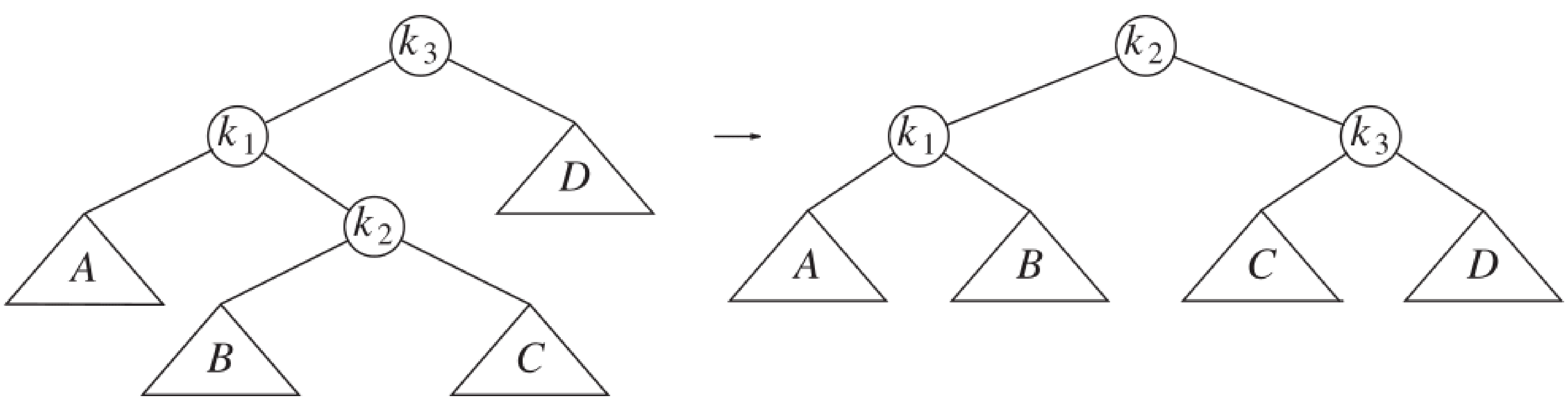
```
/**
 * Rotate binary tree node with left child.
 * For AVL trees, this is a single rotation for case 1.
 * Update heights, then set new root.
 */
void rotateWithLeftChild( AvlNode * & k2 )
{
    AvlNode *k1 = k2->left;
    k2->left = k1->right;
    k1->right = k2;
    k2->height = max( height( k2->left ), height( k2->right ) ) + 1;
    k1->height = max( height( k1->left ), k2->height ) + 1;
    k2 = k1;
}
```



AVL Trees

- implementation
  - double rotation

```
/**
 * Double rotate binary tree node: first left child
 * with its right child; then node k3 with new left child.
 * For AVL trees, this is a double rotation for case 2.
 * Update heights, then set new root.
 */
void doubleWithLeftChild( AvlNode * & k3 )
{
    rotateWithRightChild( k3->left );
    rotateWithLeftChild( k3 );
}
```



AVL Trees

- implementation
  - deletion

```
/**
 * Internal method to remove from a subtree.
 * x is the item to remove.
 * t is the node that roots the subtree.
 * Set the new root of the subtree.
 */
void remove( const Comparable & x, AvlNode * & t )
{
    if( t == nullptr )
        return;   // Item not found; do nothing

    if( x < t->element )
        remove( x, t->left );
    else if( t->element < x )
        remove( x, t->right );
    else if( t->left != nullptr && t->right != nullptr )  // Two children
    {
        t->element = findMin( t->right  )->element;
        remove( t->element, t->right );
    }
    else
    {
        AvlNode *oldNode = t;
        t = ( t->left != nullptr ) ? t->left : t->right;
        delete oldNode;
    }

     balance( t );
}
```

Splay Trees

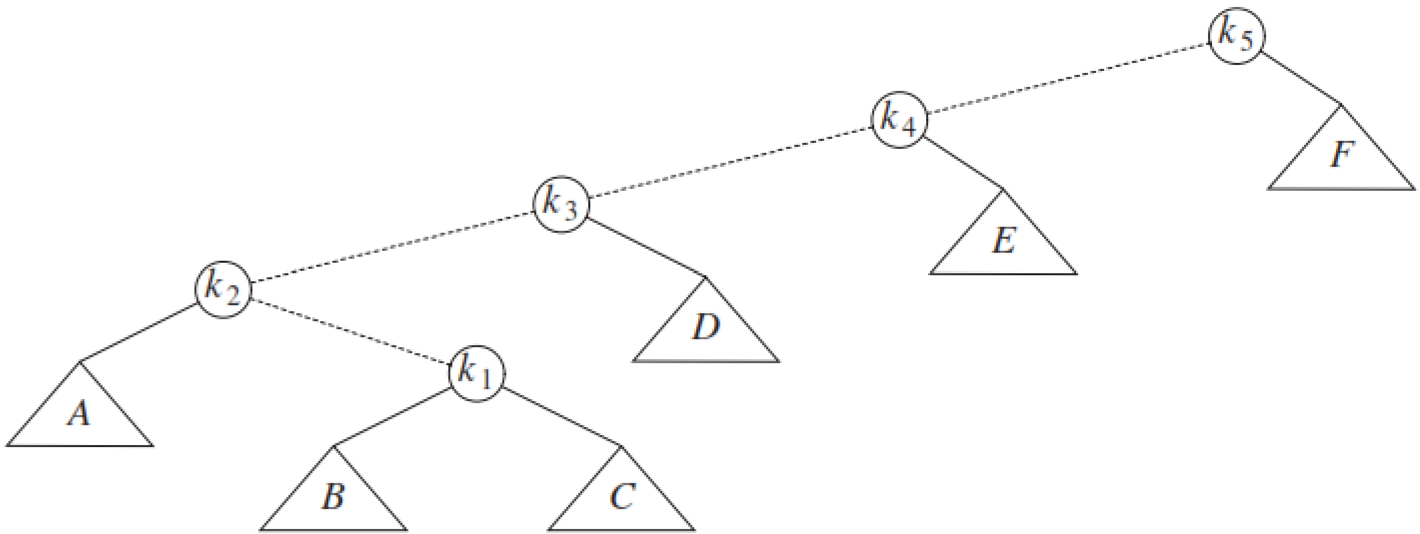
- different approach to ensuring $O(\log N)$ behavior for tree operations (searches, insertions, and deletions).
- worst case
  - splay trees operations may take N time
- however, splay trees make slow operations infrequently
  - guarantee that M consecutive operations (insertions or deletions) requires at most $O(M \log N)$, so, on average, operations are $O(\log N)$
  - $O(\log N)$ is an amortized complexity
    - derivation is complex
  - common for binary search trees to have a sustained sequence of bad accesses

Splay Trees

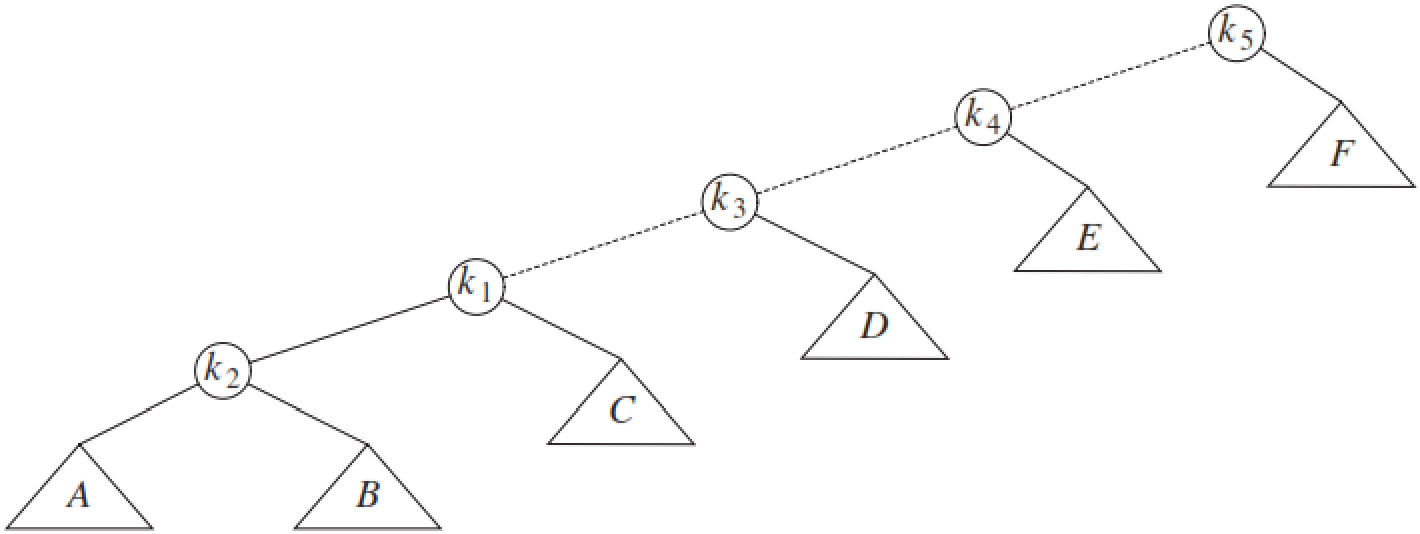
- basic idea: when a node is accessed, it is moved to the top of the tree, with the thought that we might want to revisit recently accessed nodes more frequently
  - use double rotations similar to AVL to move nodes to top of tree
  - along the way, more branching is introduced in the tree, which reduces the height of the tree and thus the cost of tree operations

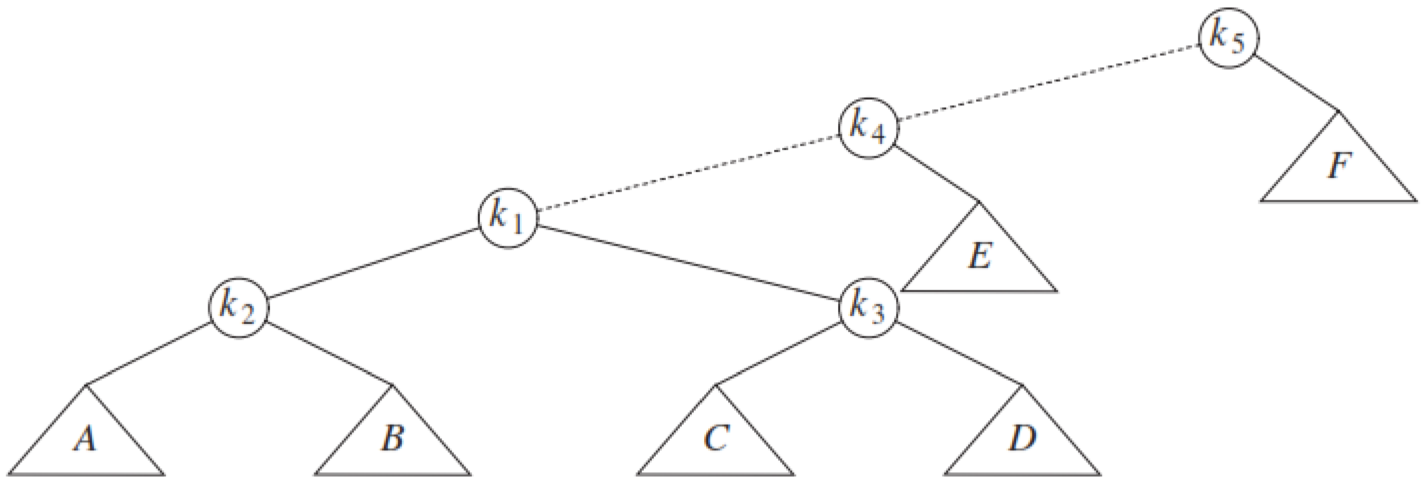
Splay Trees

- single rotations <u>don't work</u>
  - access k_1

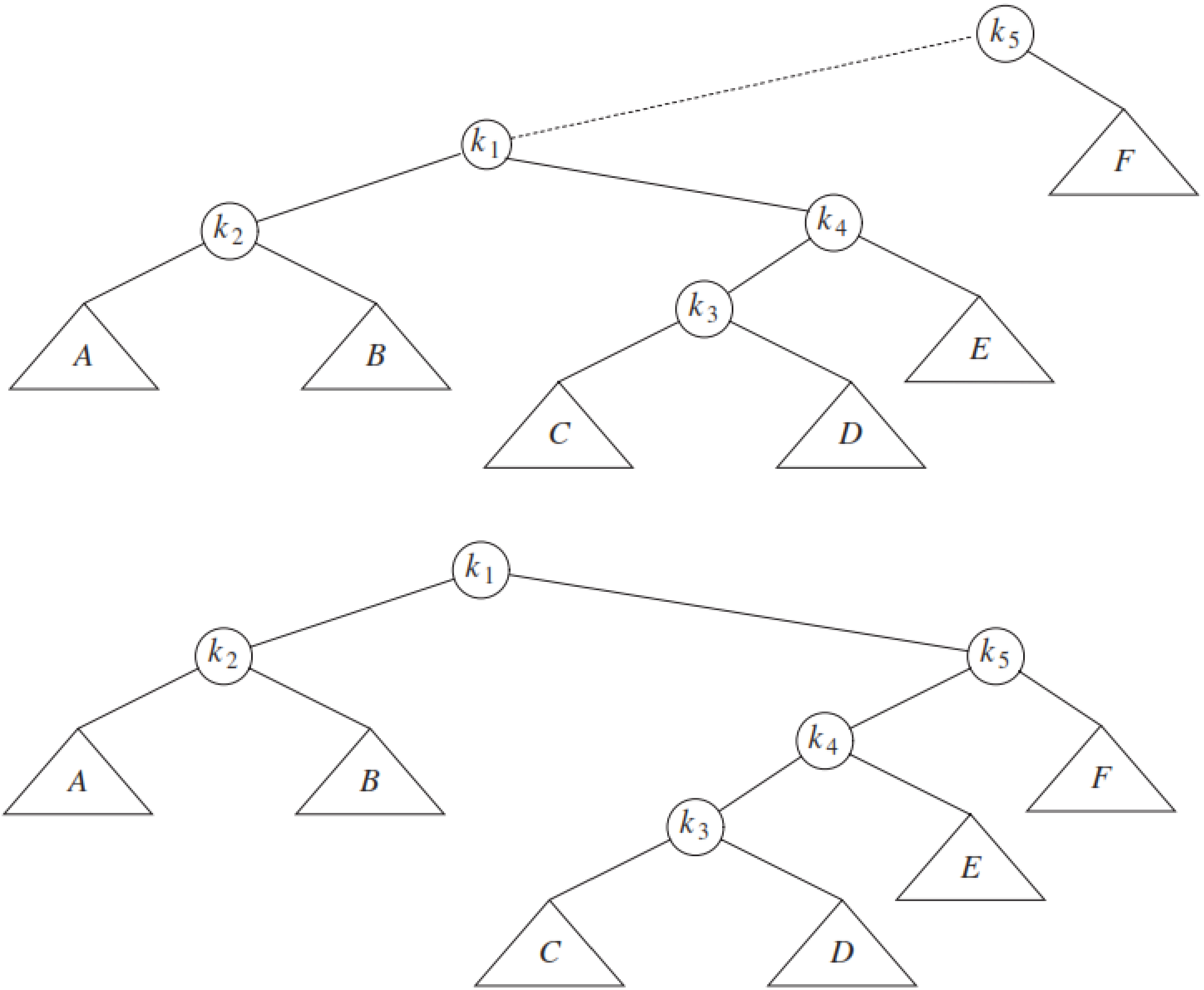


Splay Trees





Splay Trees

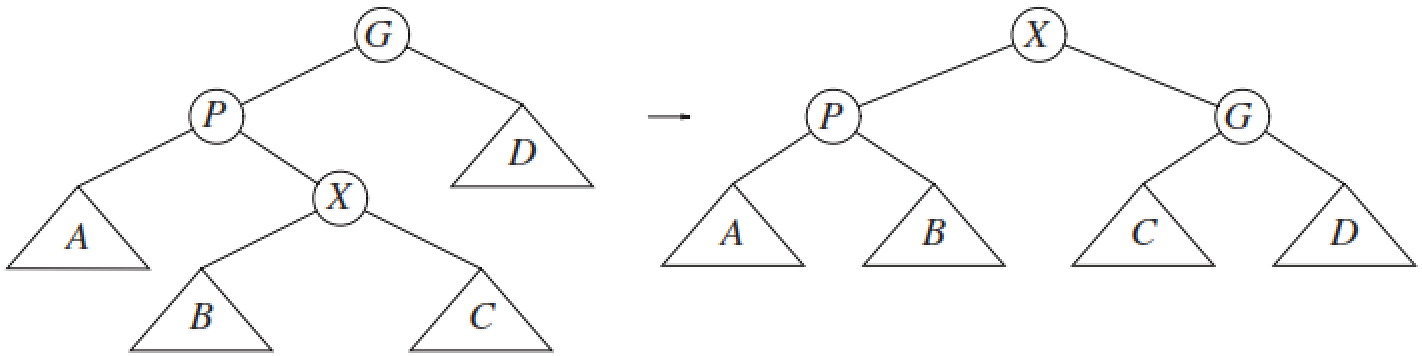


Splay Trees

- double rotations consider parent and grandparent of accessed node
  - zig: single branch (in one direction)
  - zag: secondary branch (in opposite direction)
- when the parent node is the root, a single rotation for the zig is sufficient

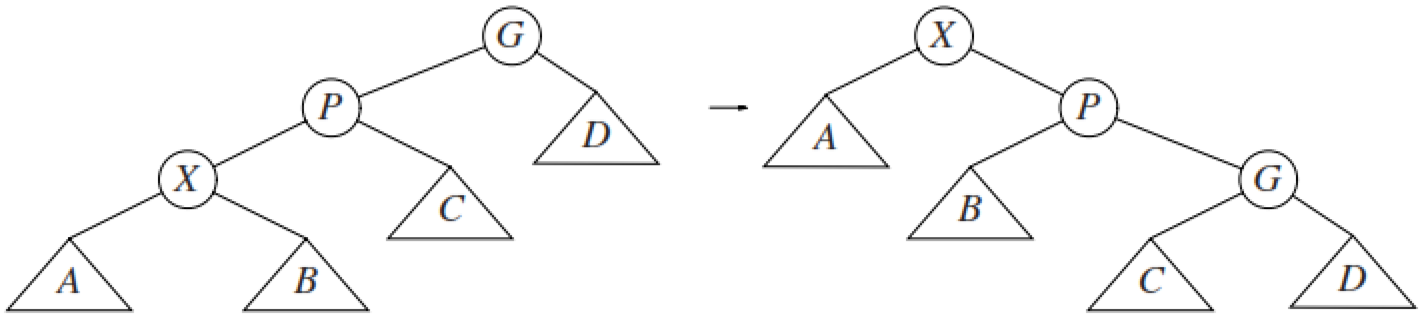
Splay Trees

- access *X*
  - zig-zag



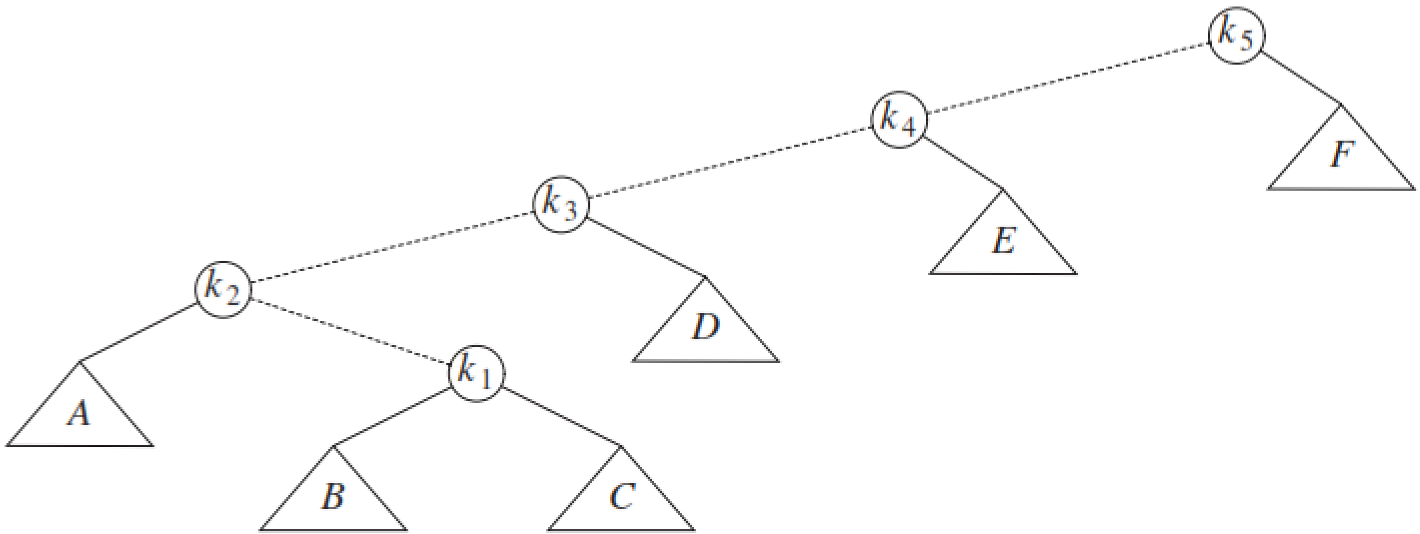
Splay Trees

- access *X*
  - zig-zig



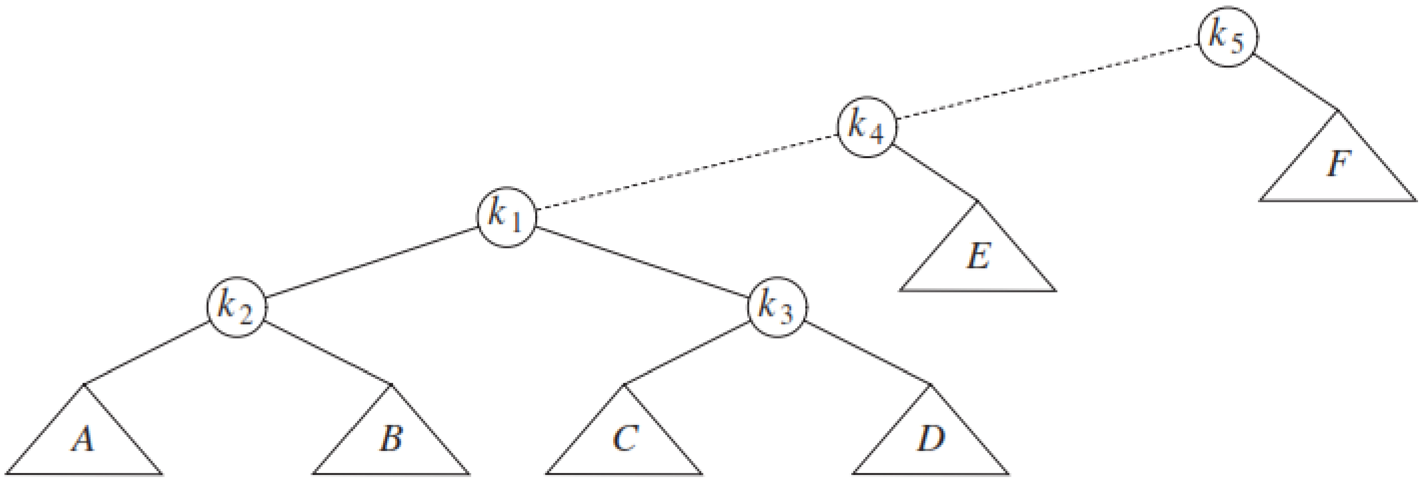
Splay Trees

- consider tree from previous example
  - access k_1
  - zig-zag



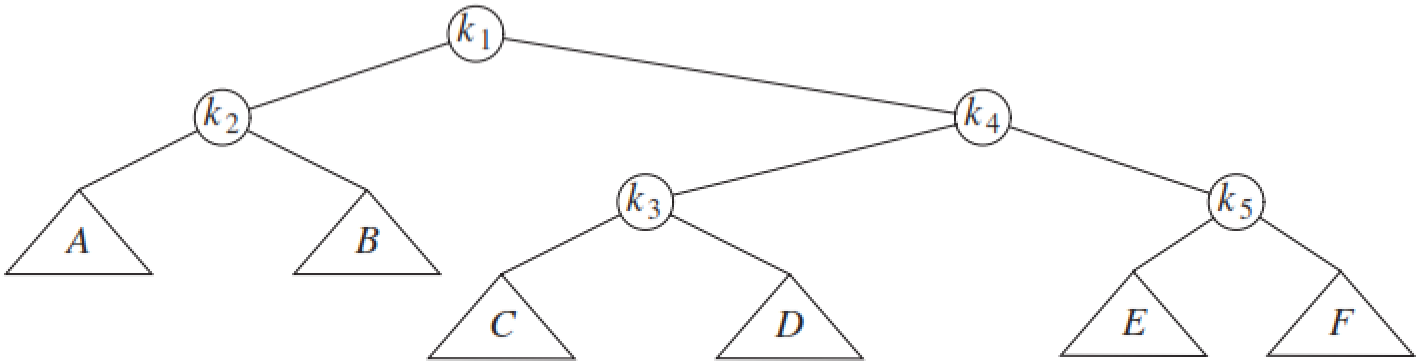
Splay Trees

- consider tree from previous example
  - access k_1
  - zig-zig



Splay Trees

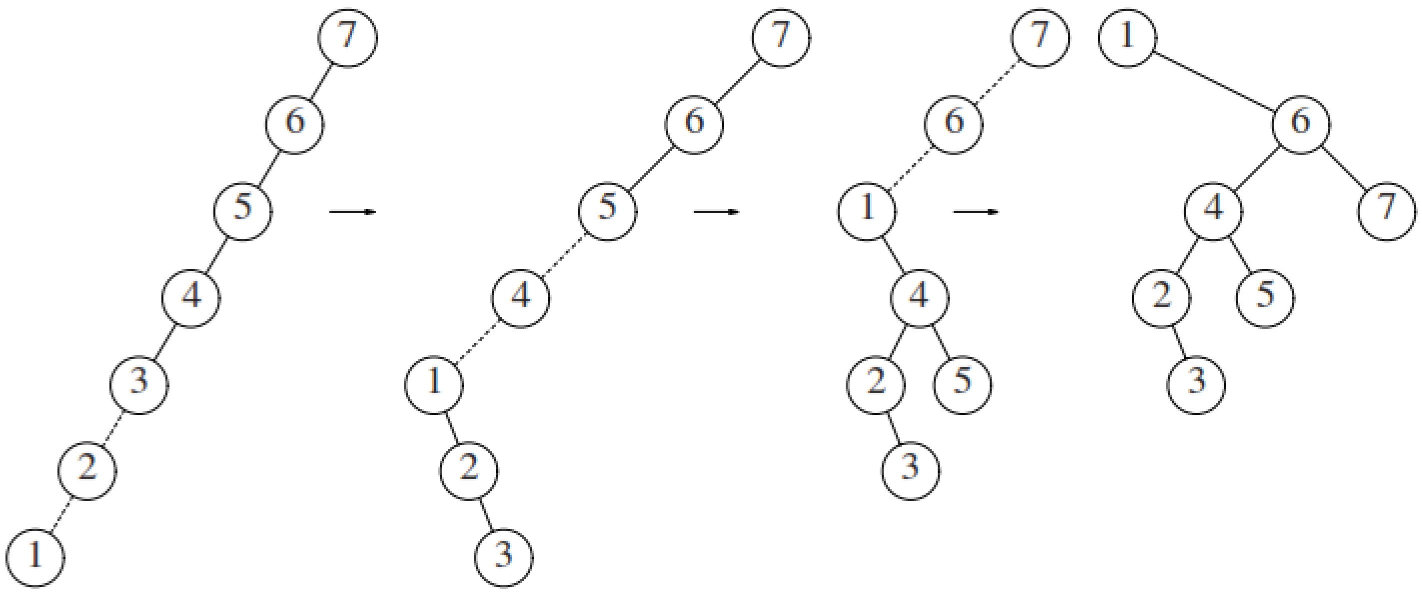
- consider tree from previous example
  - access k_1



  - k_1 is now at the root
  - final tree has <u>halved</u> the distance of most nodes on the access path to the root

Splay Trees

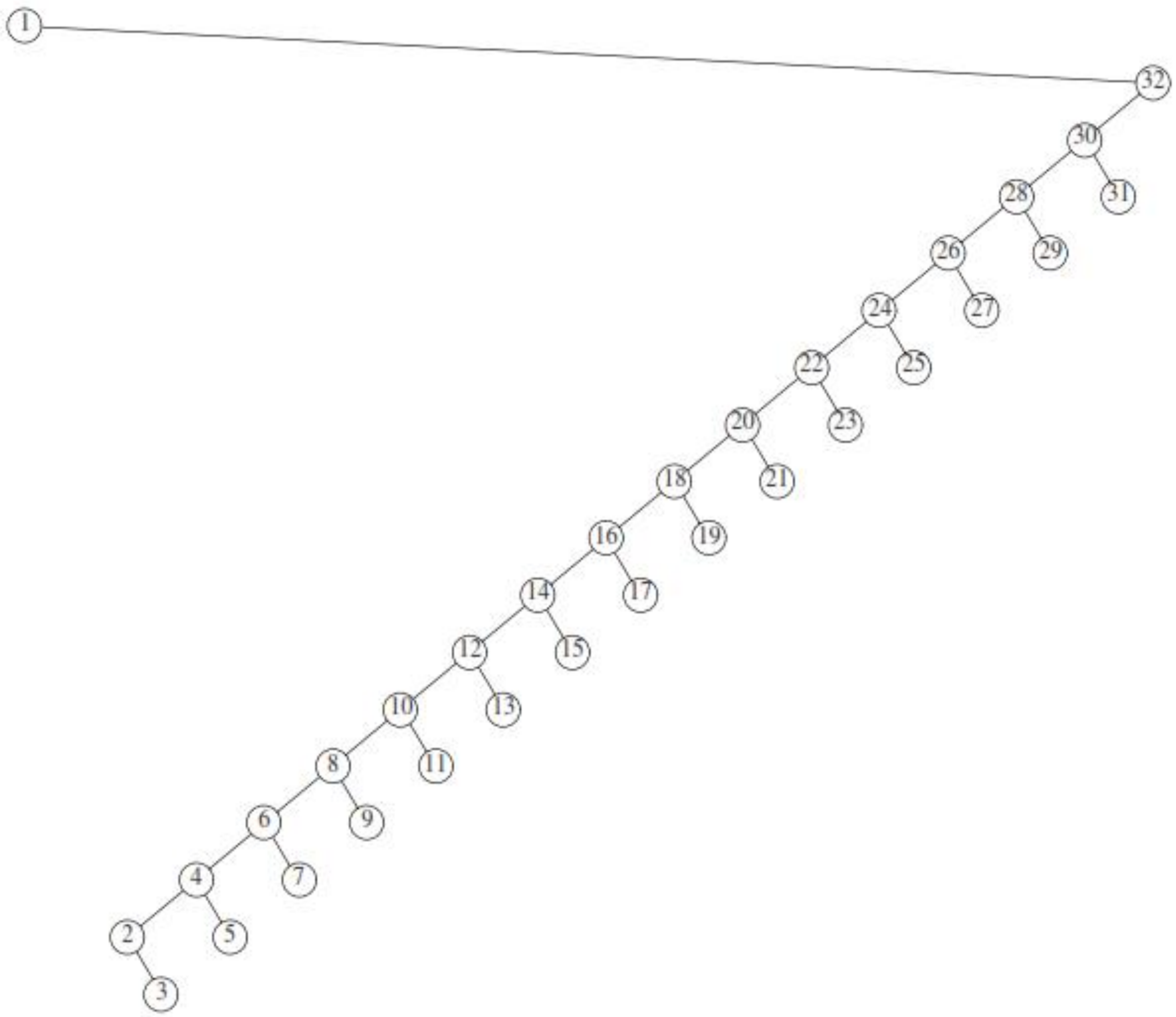
- example 2
  - access 1



  - tree starts as worst case and results in much better structure for performance

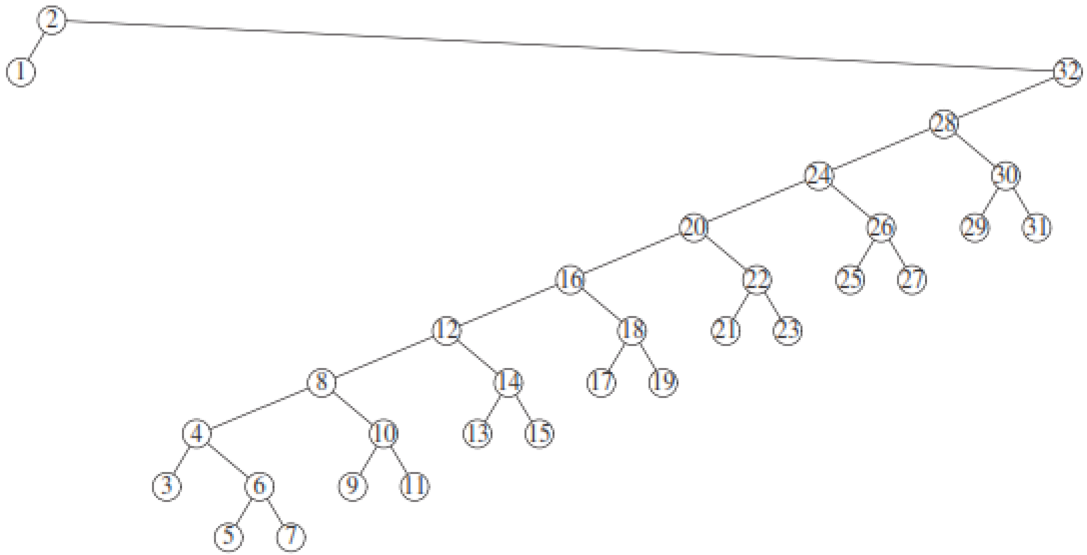
Splay Trees

– example 3: tree with only left children – access 1



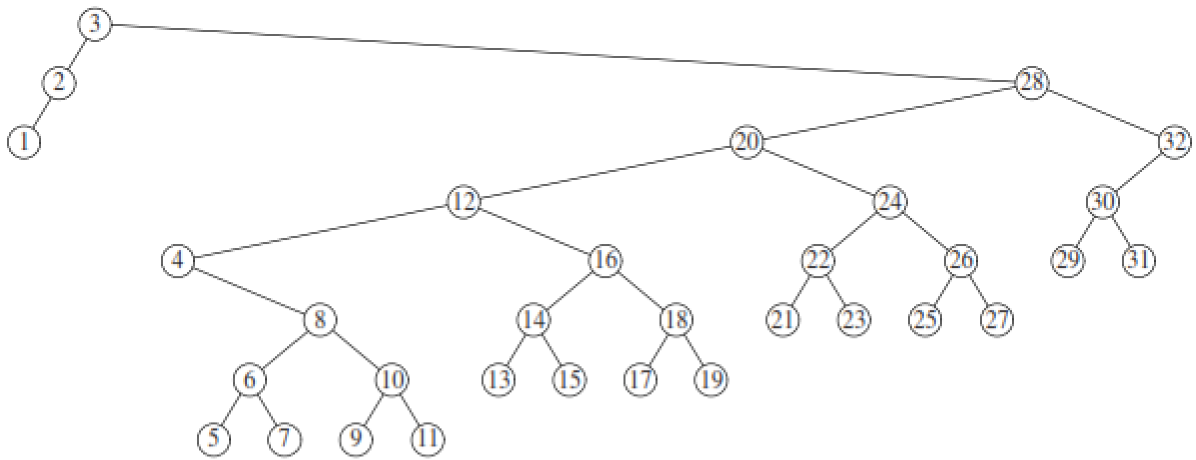
Splay Trees

- example 3: tree with only left children – access 2



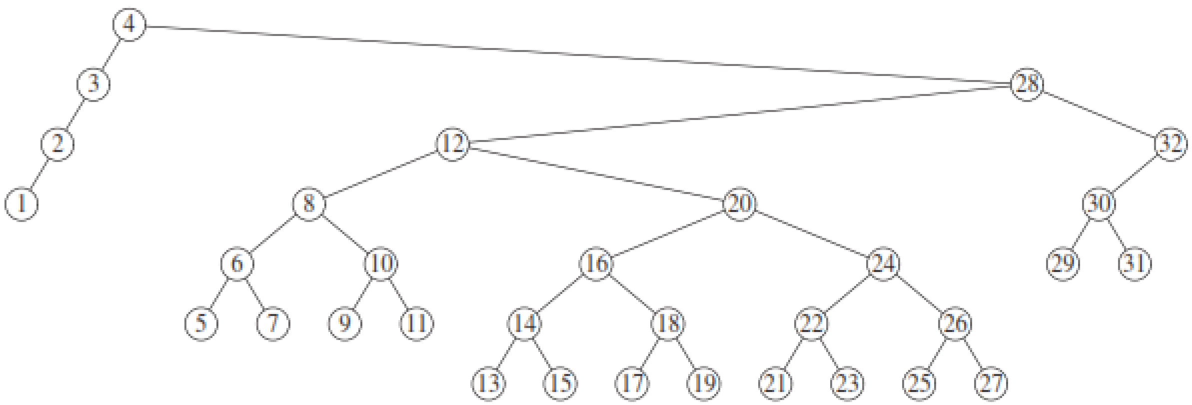
Splay Trees

- example 3: tree with only left children – access 3



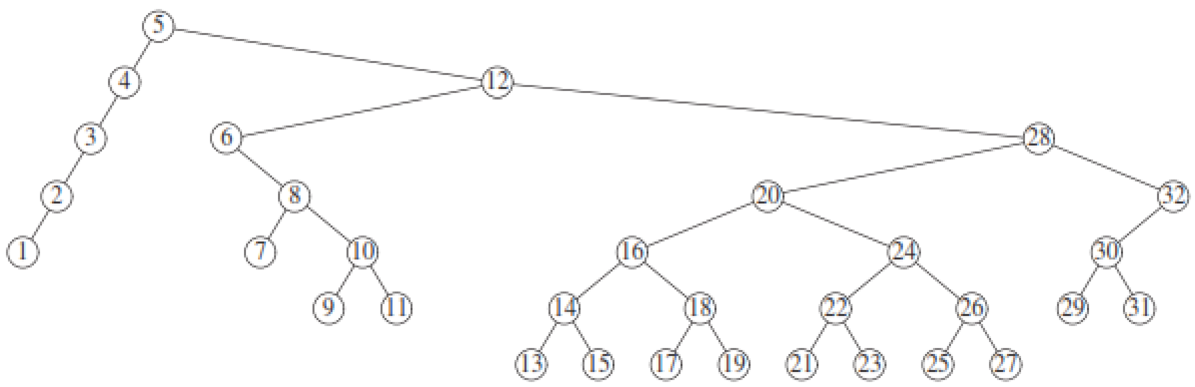
Splay Trees

- example 3: tree with only left children – access 4



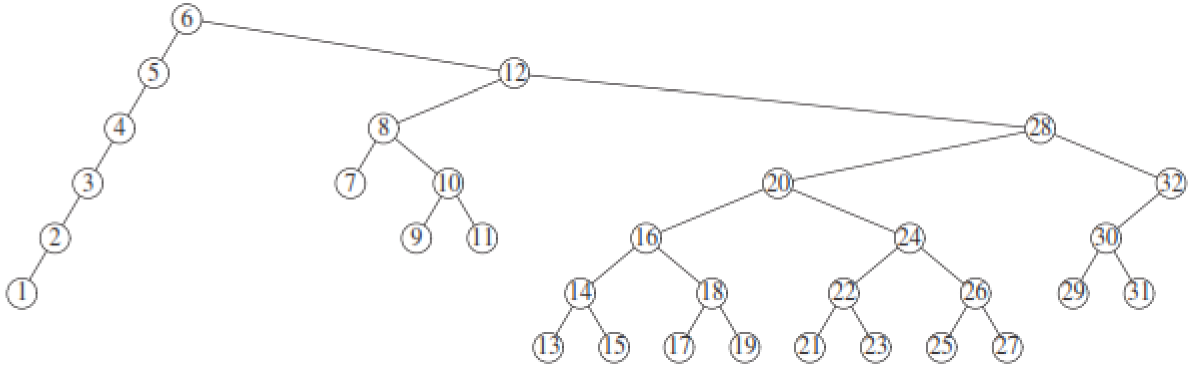
Splay Trees

- example 3: tree with only left children – access 5



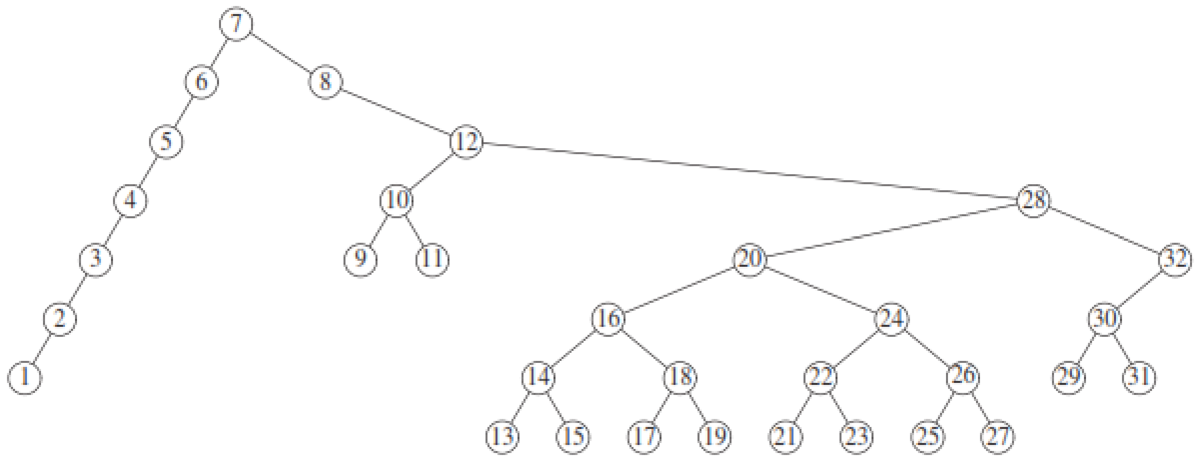
Splay Trees

- example 3: tree with only left children – access 6



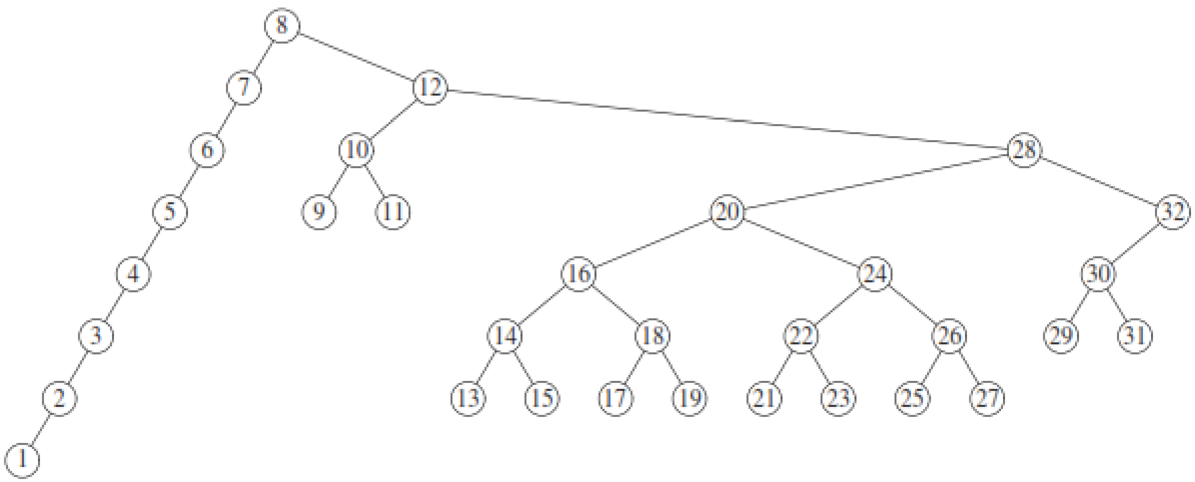
Splay Trees

- example 3: tree with only left children – access 7



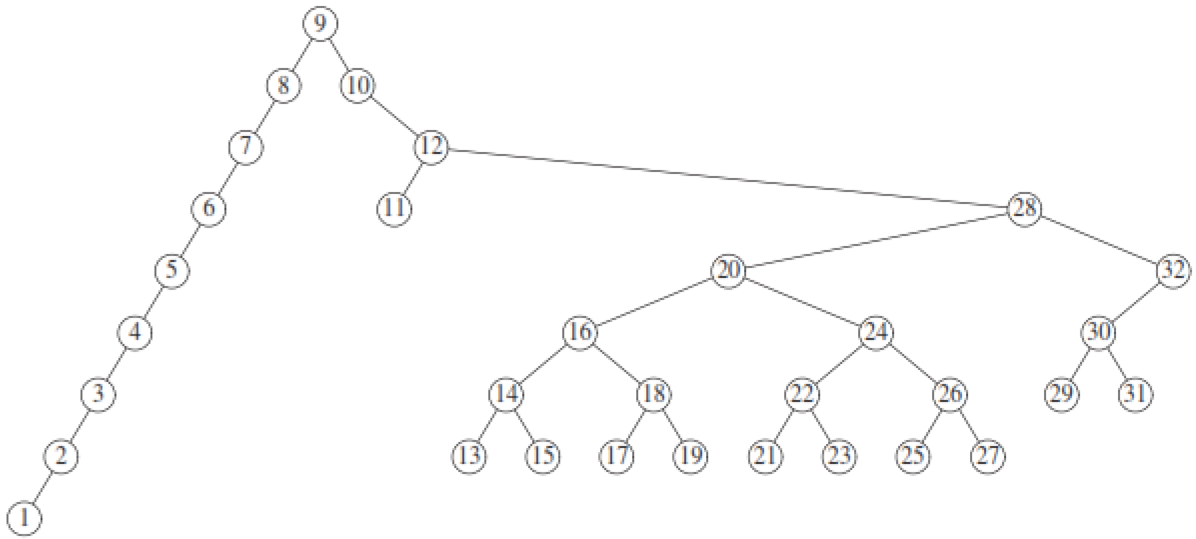
Splay Trees

- example 3: tree with only left children – access 8



Splay Trees

– example 3: tree with only left children – access 9



Splay Trees

- deleting nodes
  - first, access the node, which moves it to the root of the tree
    - let T_L and T_R be the left and right subtrees of the new root
  - find e, the largest element of T_L
  - rotate e to the root of T_L
  - since e is the largest element of T_L, it will have no right child, so we can attach T_R there
    - rather than the largest element of T_L, we could use the smallest element of T_R and modify T_R

Top-Down Splay Trees

- previous method requires traversal from root down to node, then a bottom-up traversal to implement the splaying
  - can be accomplished by maintaining <u>parent</u> links
  - or by storing <u>access path</u> on the stack
  - both methods require substantial <u>overhead</u>
  - both must handle a variety of special cases

Top-Down Splay Trees

- instead, perform rotations on <u>initial access path</u>
  - result is faster
  - uses extra space $O(1)$
  - retains amortized time bound of $O(\log N)$

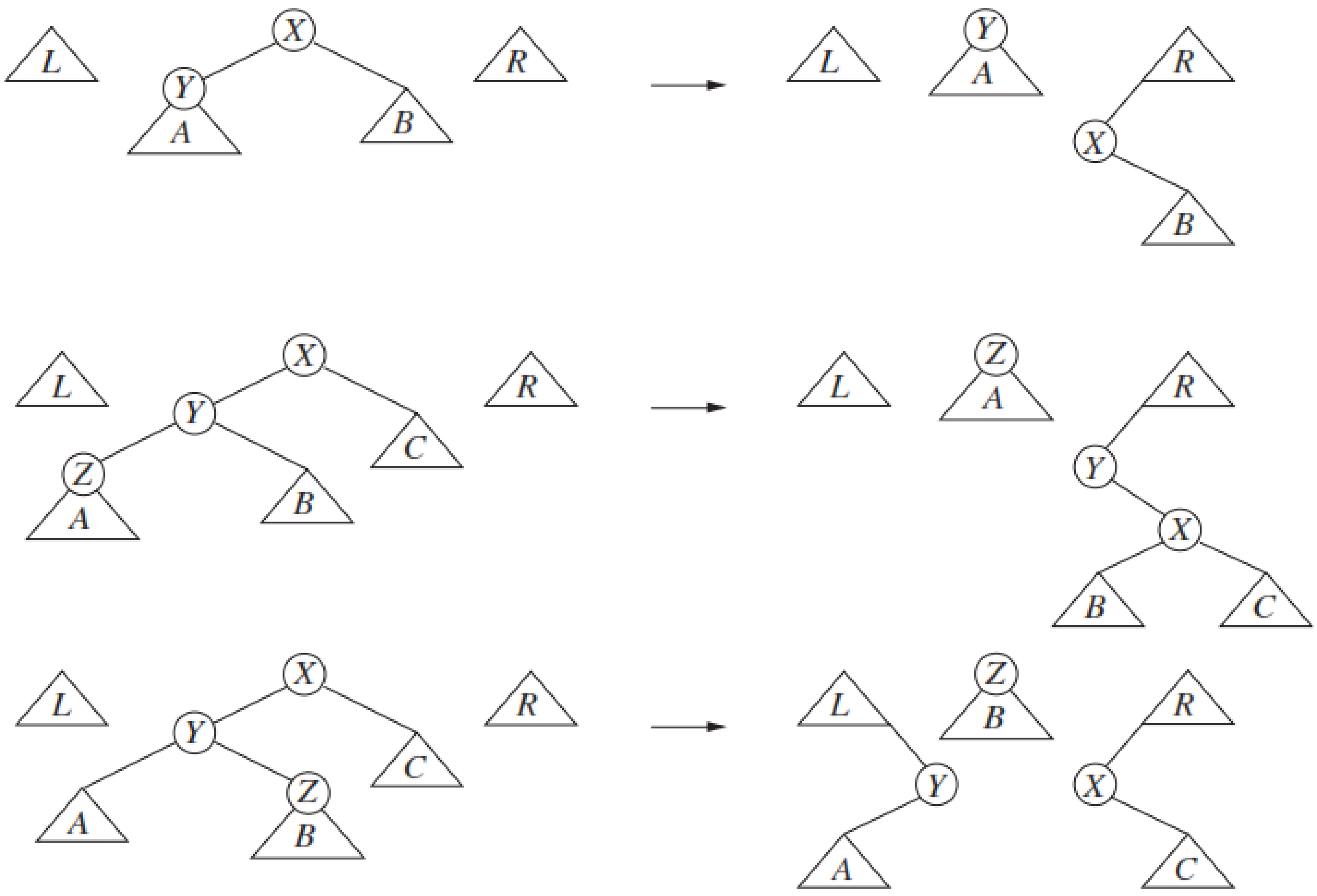
Top-Down Splay Trees

- suppose we wish to access key i
- during the access and concurrent splaying operation, the tree is broken into <u>three</u> parts
  - a left tree, which contains all the keys from the original tree known at the time to be <u>less</u> than i
  - a right tree, which contains all the keys from the original tree known at the time to be <u>greater</u> than i
  - a middle tree, which consists of the subtree of the original tree rooted at the current node on the access path

- initially, the left and right trees are empty and the middle tree is the entire tree
- at each step we tack bits of the middle tree onto the left and right trees

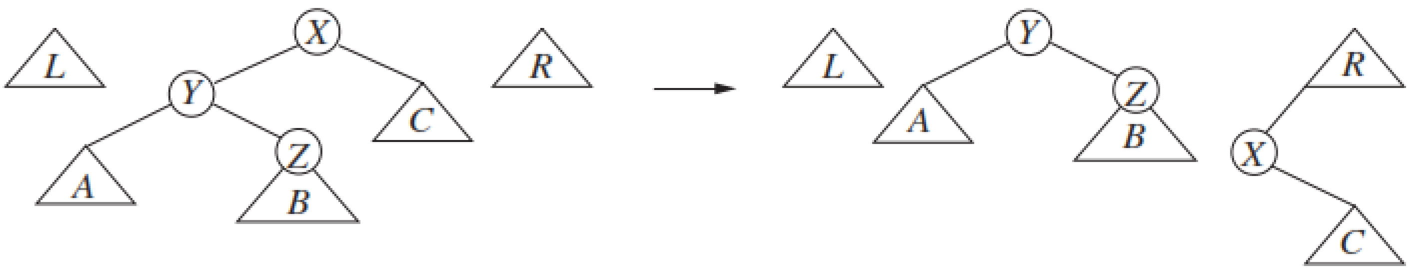
Top-Down Splay Trees

- rotations for zig, zig-zig, and zig-zag cases



Top-Down Splay Trees

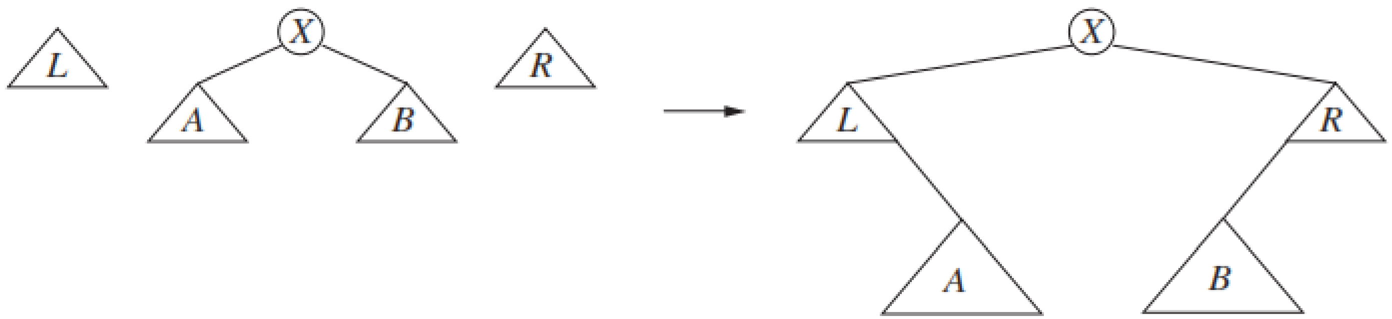
- zig-zag case can be simplified to just a <u>zig</u> since no rotations are performed
- instead of making Z the root, we make Y the root



- simplifies coding, but only descends one level
  - requires more <u>iterations</u>

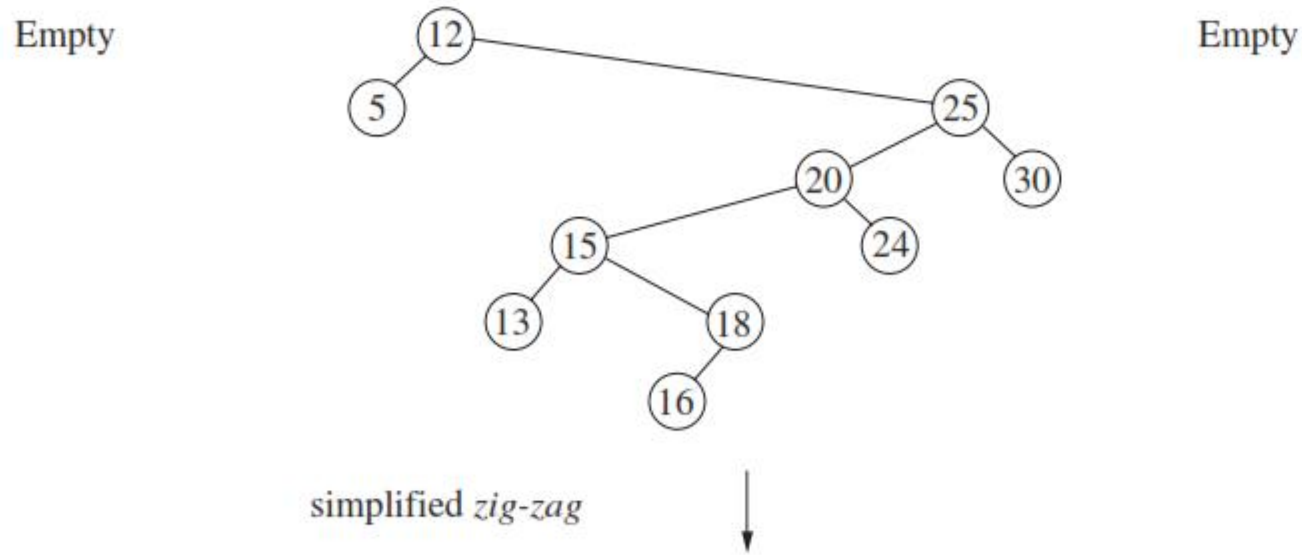
Top-Down Splay Trees

- after final splaying



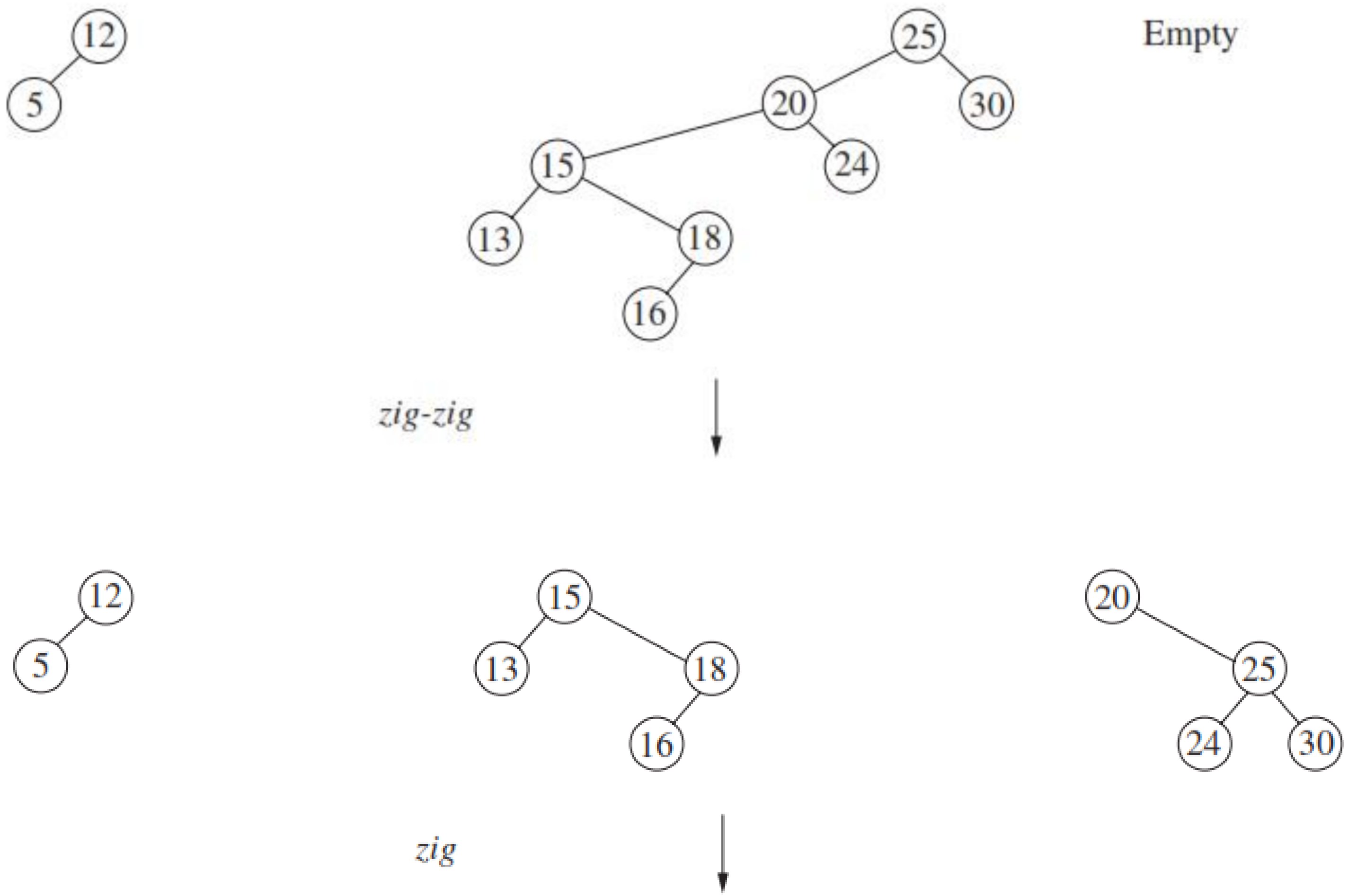
Top-Down Splay Trees

- example: access 19



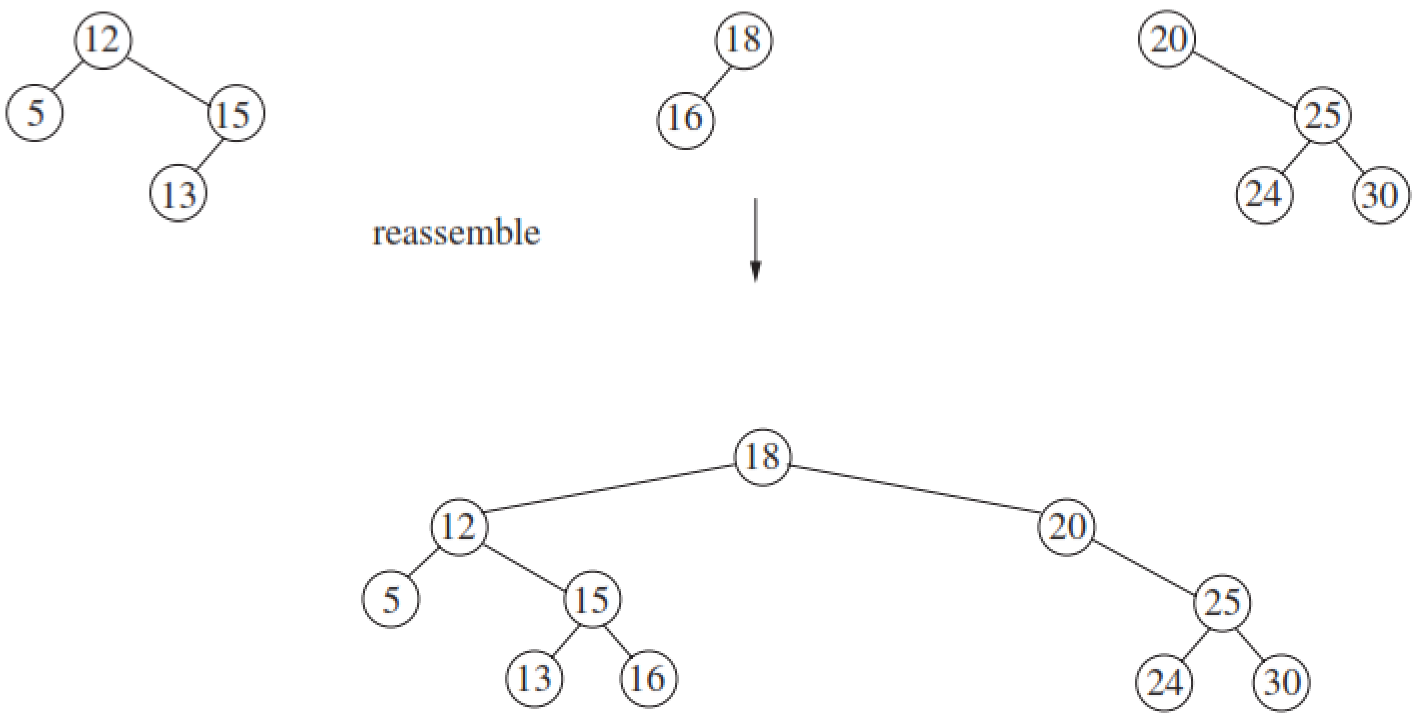
Top-Down Splay Trees

- example: access 19



Top-Down Splay Trees

- example: access 19



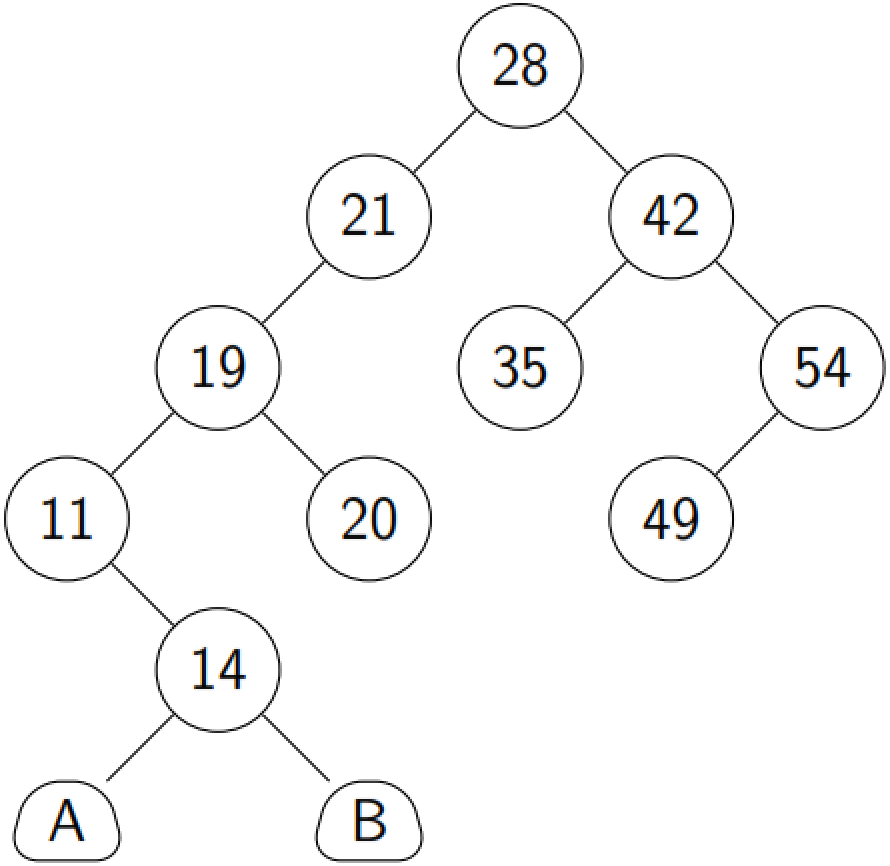
Top-Down Splay Trees

- use a header to hold the roots of the left and right subtrees
  - left pointer will contain root of right subtree
  - right pointer will contain root of left subtree
  - easy to reconstruct at end of splaying

Top-Down Splay Trees

– example 2: access 14

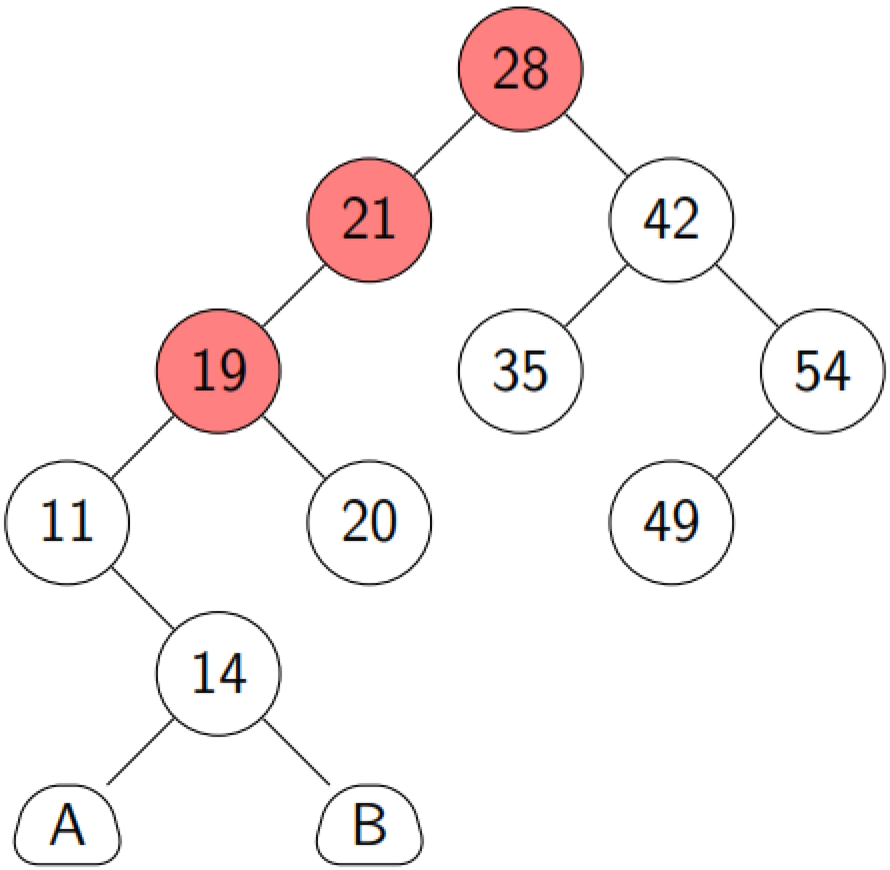
empty
tree



empty
tree

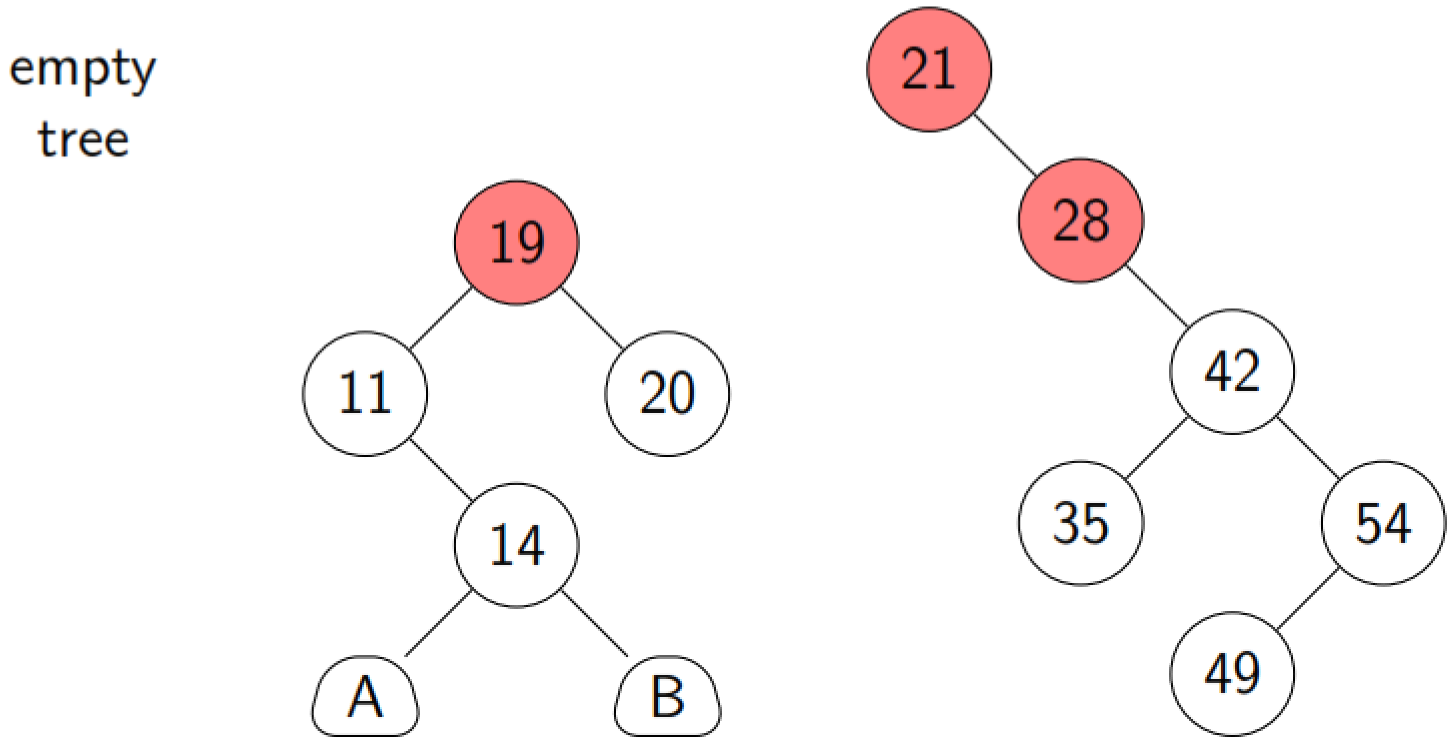
Top-Down Splay Trees

- example 2: access 14
- start at root and look down two nodes along path to 14



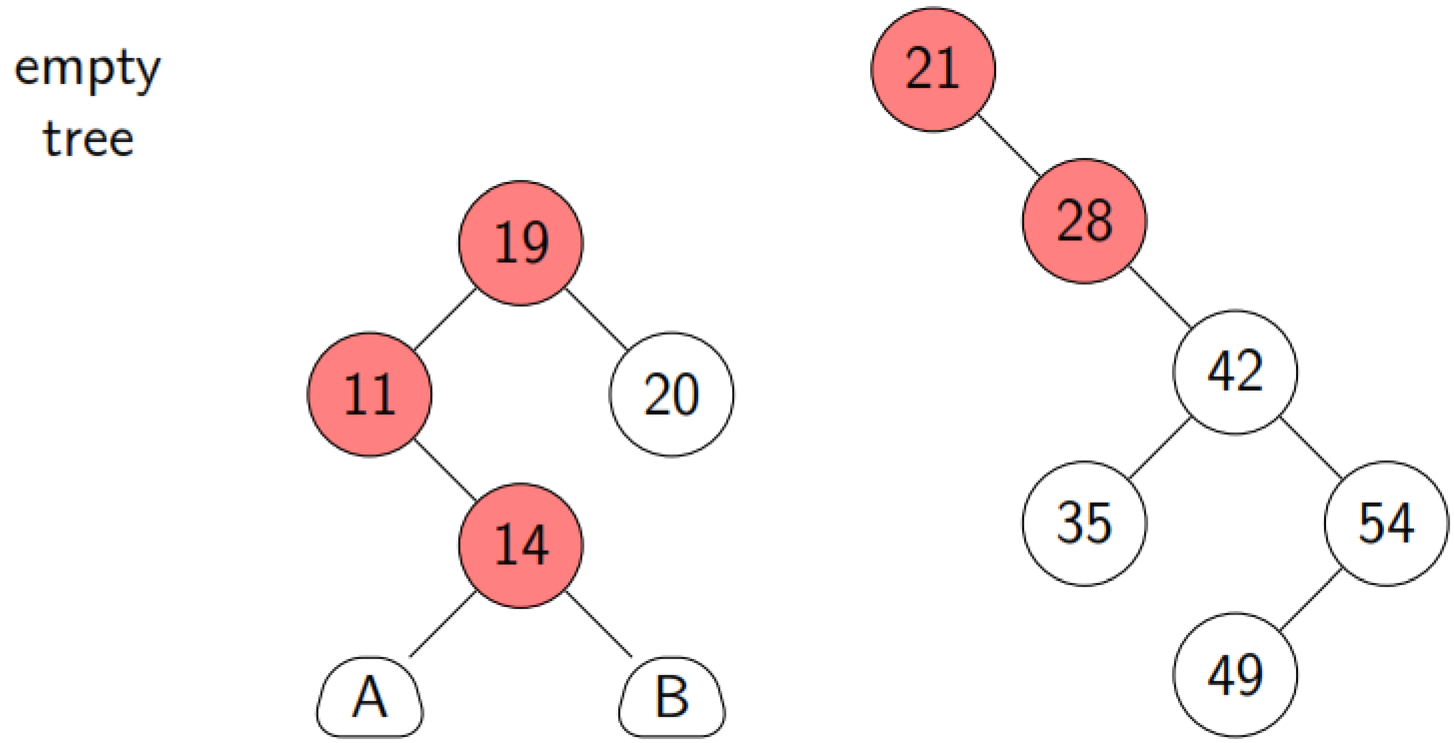
Top-Down Splay Trees

- example 2: access 14



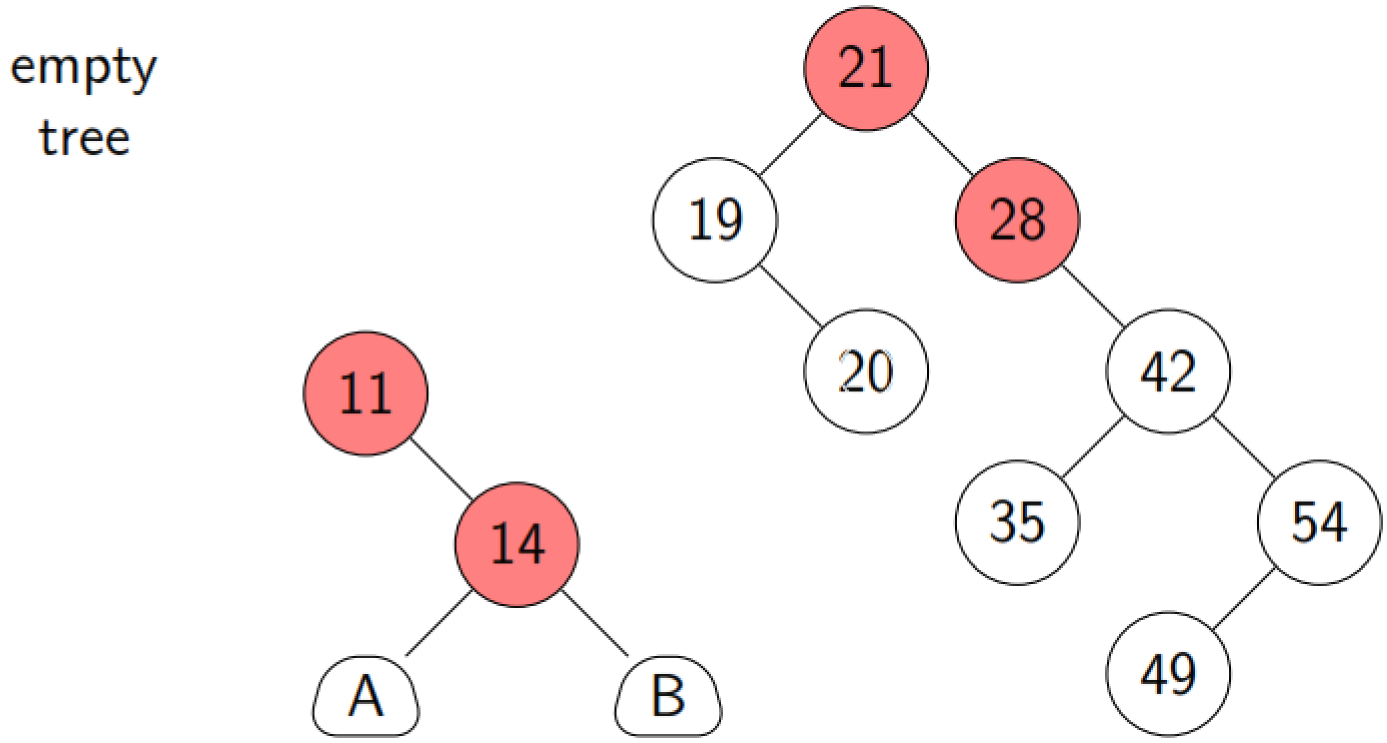
Top-Down Splay Trees

- example 2: access 14
- continuing down the tree, this is a <u>zig-zag</u> condition



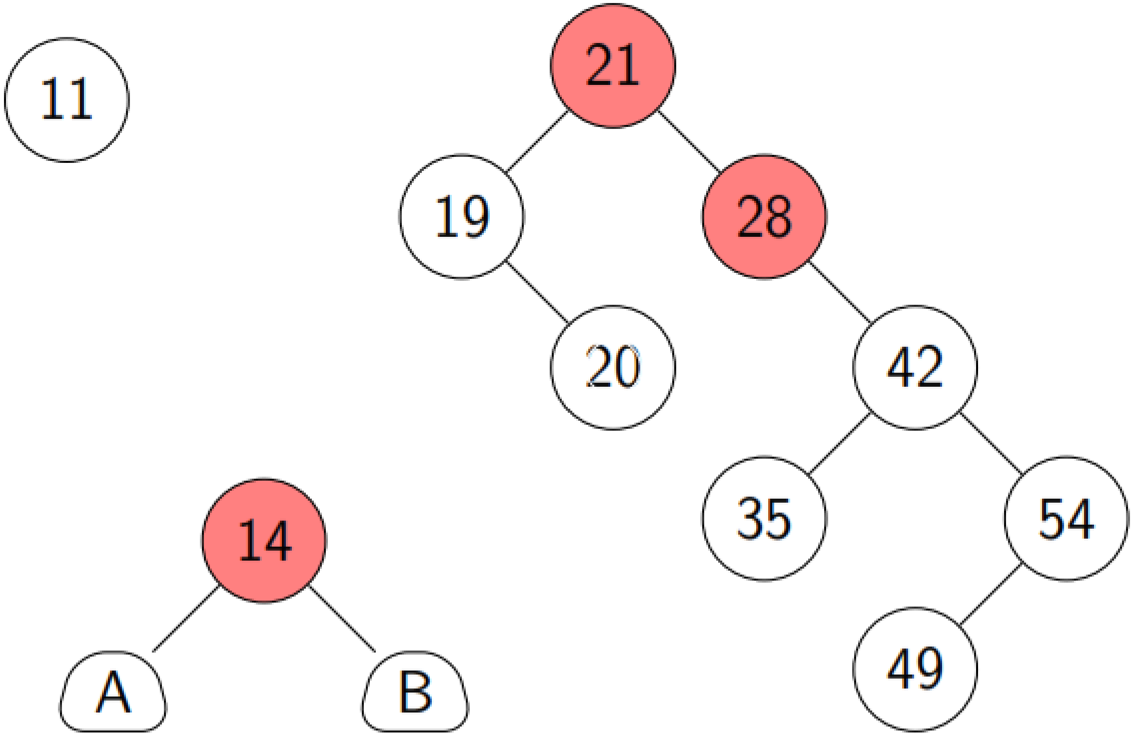
Top-Down Splay Trees

- example 2: access 14
- tree is reconfigured



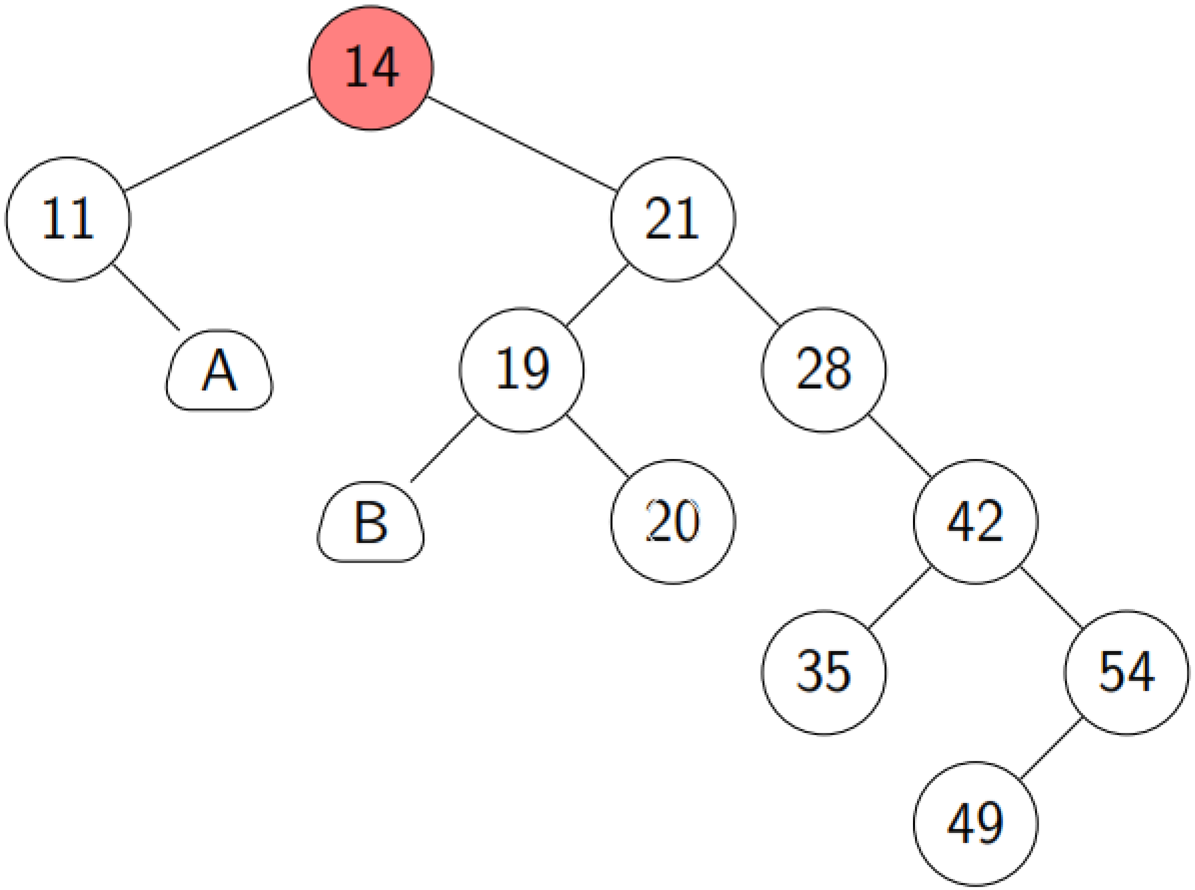
Top-Down Splay Trees

- example 2: access 14
- simple zig



Top-Down Splay Trees

- example 2: access 14
- move accessed node to root and reassemble tree



B-Trees

- B-trees were developed in the late 1960s by Rudolf Bayer and Edward McCreight:

  R. Bayer and E. McCreight, *Organization and maintenance of large ordered indexes*, Acta Informatica vol. 1, no. 3 (1972), pp. 173-189.

  - originally motivated by applications in <u>databases</u>
  - B-trees shown here really B+ tree

B-Trees

- thus far, we have frequently treated the key as if it were the <u>data</u> being stored, but that is rarely the case
- example: student records in Banner
  - most effective search key is W&M ID (e.g., 930…) since it is unique
  - the record (value) associated with each key contains much more information
    - Student Information
    - Student Academic Transcript
    - Student Active Registrations
    - Student Schedule
    - Student E-mail Address
    - Student Address and Phones ...

B-Trees

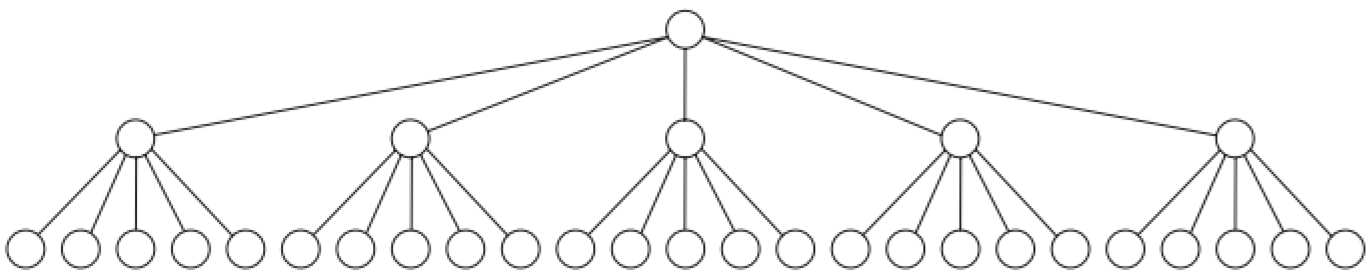
- B-trees are particularly useful when we cannot fit all of our data in memory, but have to perform reads and writes from secondary storage (e.g., disk drives)
  - disk accesses incredibly expensive, relatively speaking
  - consider a disk drive that rotates at 7200 rpm
    - the rotational speed plays a role in retrieval time; for a 7200 rpm disk, each revolution takes 60/7200 = 1/120 s, or about 8.3 ms
    - a typical seek time (the time for the disk head to move to the location where data will be read or written) for 7200 rpm disks is around 9 ms
    - this means we can perform 100-120 random disk accesses per second

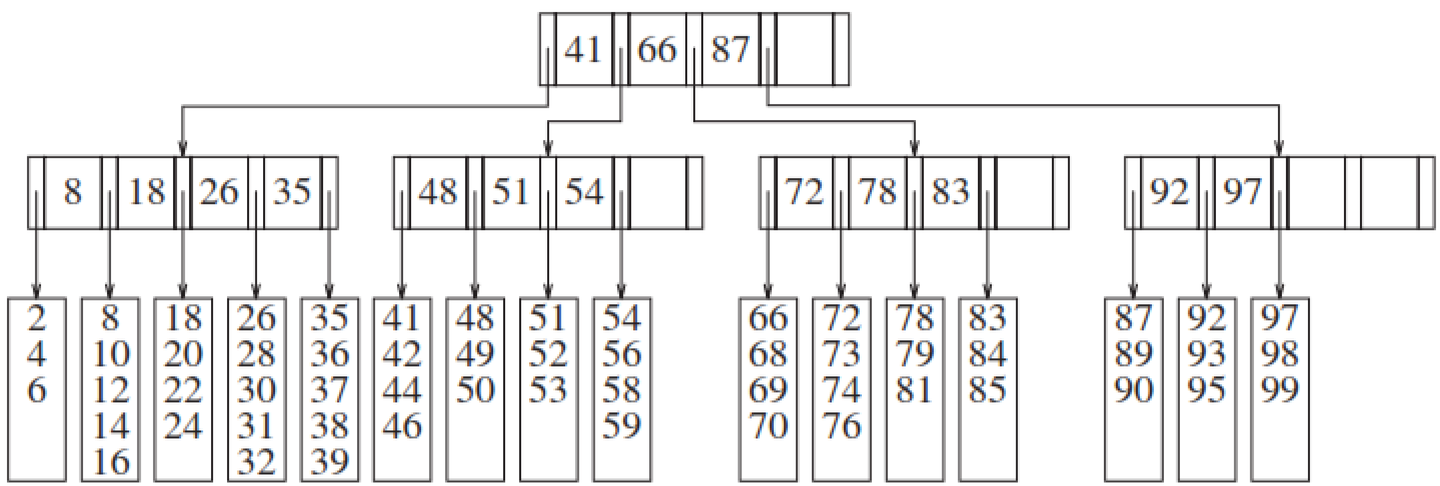
B-Trees

- meanwhile, our CPU can perform > 1,000,000,000 operations per second
- suppose we have a database with N = 10,000,000 entries that we organize in a tree
  - in an AVL tree, a worst-case search requires 1:44 lg N ≈ 33 disk accesses
  - at 9 ms per access, this requires about 300 ms, so on average we can perform less than 4 searches per second
  - we would expect 1000 worst-case searches to take 300,000 ms = 300 s, or about 5 minutes
  - in this application, search trees with height $\lg N$ are still too high!

B-Trees

- height can be reduced if we allow more <u>branching</u>
  - binary search trees only allow 2-way branching
  - example: 5-ary 31-node tree with height 3





B-Trees

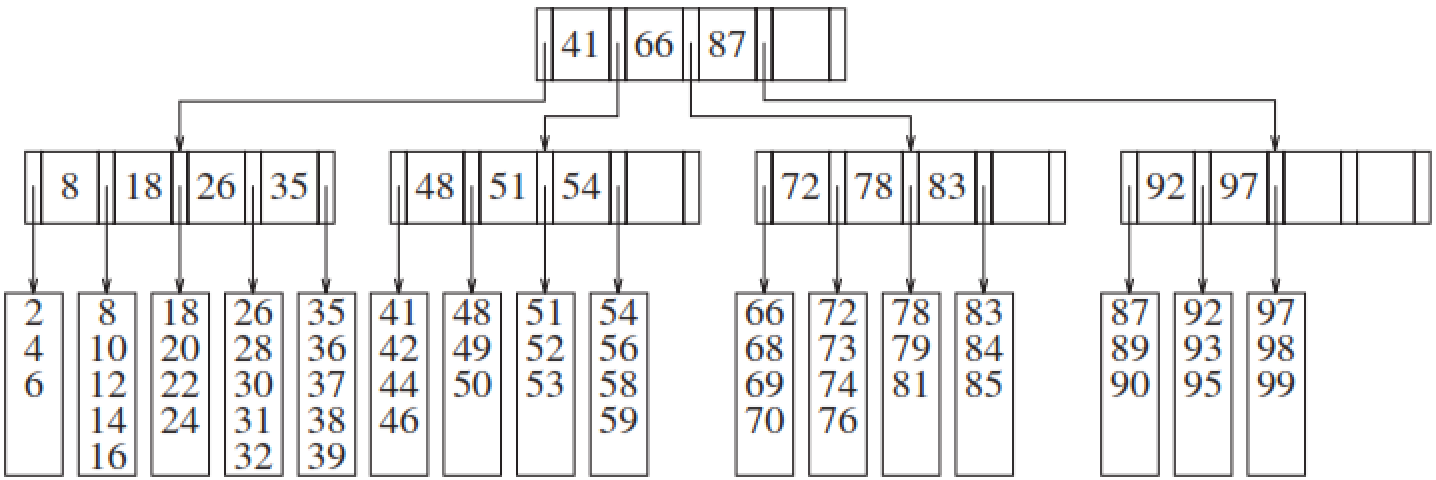
- B-tree of order M is an M-ary tree with the following properties
  1. data items are stored at leaves
  2. nonleaf nodes (internal nodes) store up to $M - 1$ keys to guide the searching: key i represents the smallest key in subtree $i + 1$
  3. root is either a leaf or has between two and M children
  4. all nonleaf nodes (except the root) have between $\lceil M/2 \rceil$ and M children
  5. all leaves are at the same depth and have between $\lceil L/2 \rceil$ and L data items, for some L

B-Trees

- examples
  - for $M = 2$, there are between $\lceil 2/2 \rceil$ = 1 to 2 children
  - for $M = 3$, there are between $\lceil 3/2 \rceil$ = 2 to 3 children
  - for $M = 4$, there are between $\lceil 4/2 \rceil$ = 2 to 4 children
  - for $M = 5$, there are between $\lceil 5/2 \rceil$ = 3 to 5 children
  - for $M = 42$, there are between $\lceil 42/2 \rceil$ = 21 to 42 children
- requiring nodes to be <u>half full</u> guarantees that the tree will not degenerate into a simple binary search tree

B-Trees

- examples: $M = 5$



- all nonleaf nodes have between 3 and 5 children (and thus between 2 and 4 keys)
- root could have just 2 children
- here L is also 5: each leaf has between 3 and 5 data items

B-Trees

- choosing M and L
  - each node will occupy a disk block, say 8192 bytes, so we choose M and L based on the size of the items being stored
  - suppose each key uses 32 bytes and a link to another node uses 8 bytes
  - a node in a B-tree of order M has $M - 1$ keys and M links, so a node requires

$$32(M - 1) + 8M = 40M - 32 \text{ bytes}$$

  - we choose the largest M that will allow a node to fit in a block

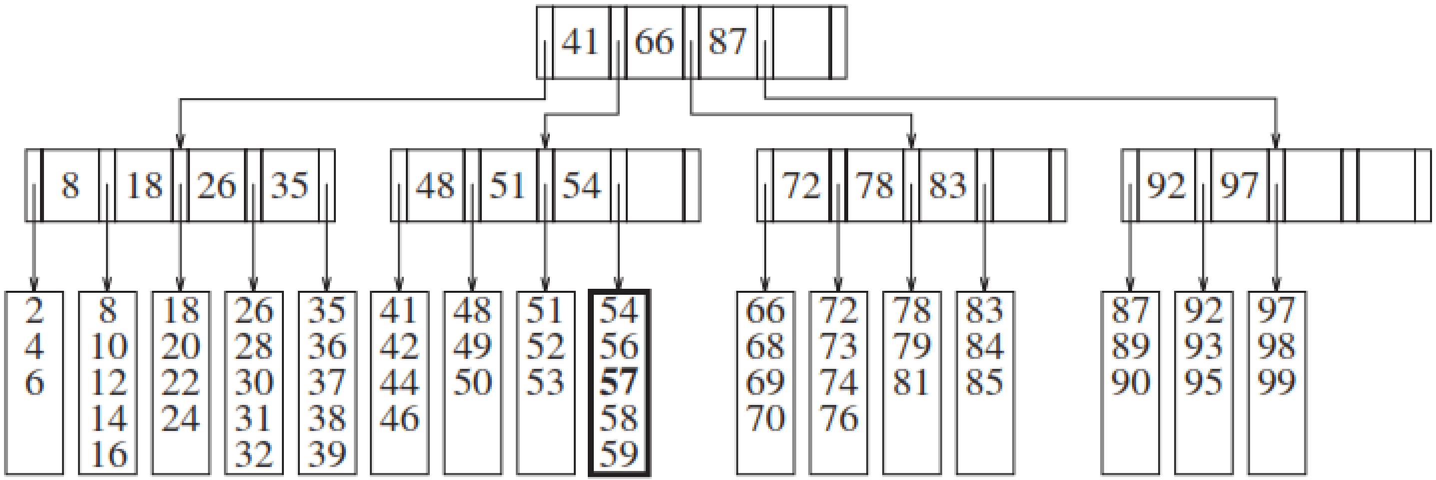
$$M = \left\lfloor \frac{8192 + 32}{40} \right\rfloor = 205$$

B-Trees

- choosing M and L (cont.)
  - if the values are each 256 bytes, then we can fit
    $$L = \left\lfloor \frac{8192}{256} \right\rfloor = 32$$
    in a single <u>block</u>
  - each <u>leaf</u> has between 16 and 32 values, and each internal node branches in at least 103 ways
  - if there are 1,000,000,000 values to store, there are at most 62,500,000 leaves
  - the leaves would be, in the worst case, on level
    $$1 + \log_{103} 62{,}500{,}000 = 5$$
    so we can find data in at most 5 disk access
  - a BST would have at least $1 + \log_2 62{,}500{,}000 = 27$ levels!

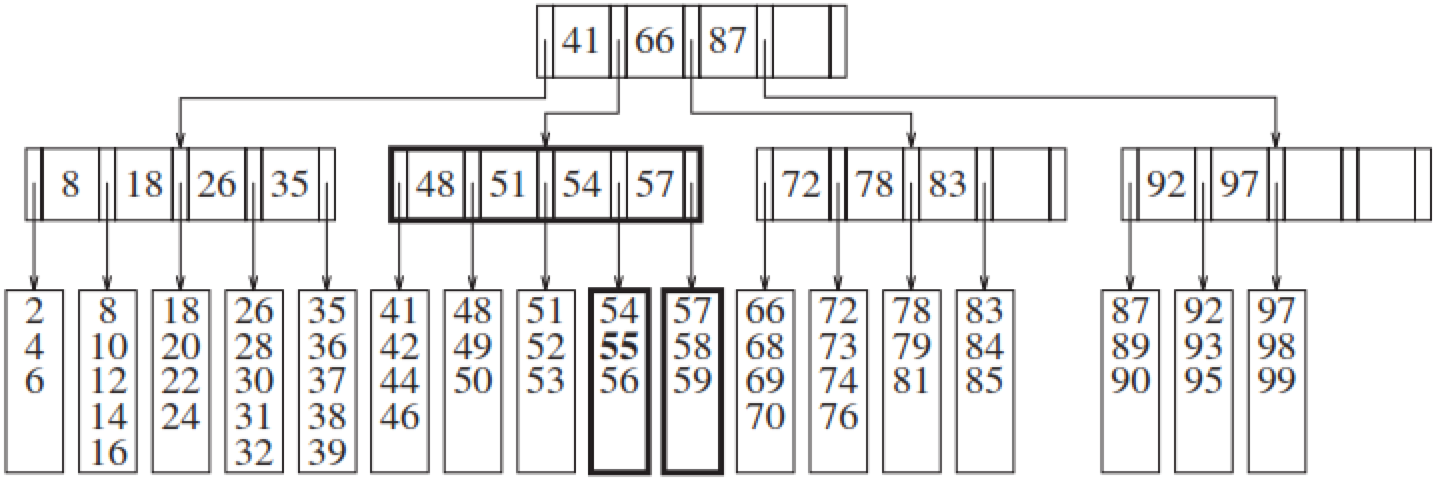
B-Trees

- insertion: easy case – insert 57
  - first, follow the search tree to the correct leaf (external node)
  - if there are fewer than L items in the leaf, insert in the correct location
    - cost: 1 disk access
- insert 55?



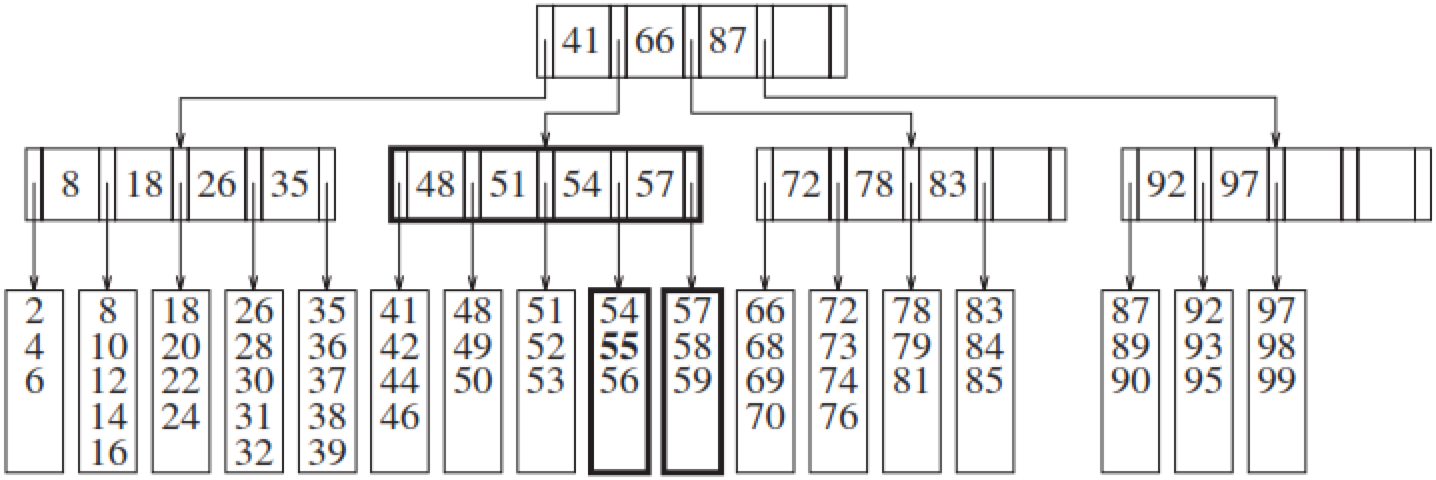
B-Trees

- insertion: splitting a leaf – insert 55
  - if there are already L items in the leaf
    - add the new item, split the node in two, and update the links in the parent node
    - cost: 3 disk accesses (one for each new node and one for the update of the parent node)



B-Trees

- insertion: splitting a leaf – insert 55 (cont.)
  - the splitting rule ensures we still have a <u>B-tree</u>: each new node has at least $\lceil L/2 \rceil$ values (e.g., if $L = 3$, there are 2 values in one node and 1 in the other, and if $L = 4$, each new node has 2 keys)

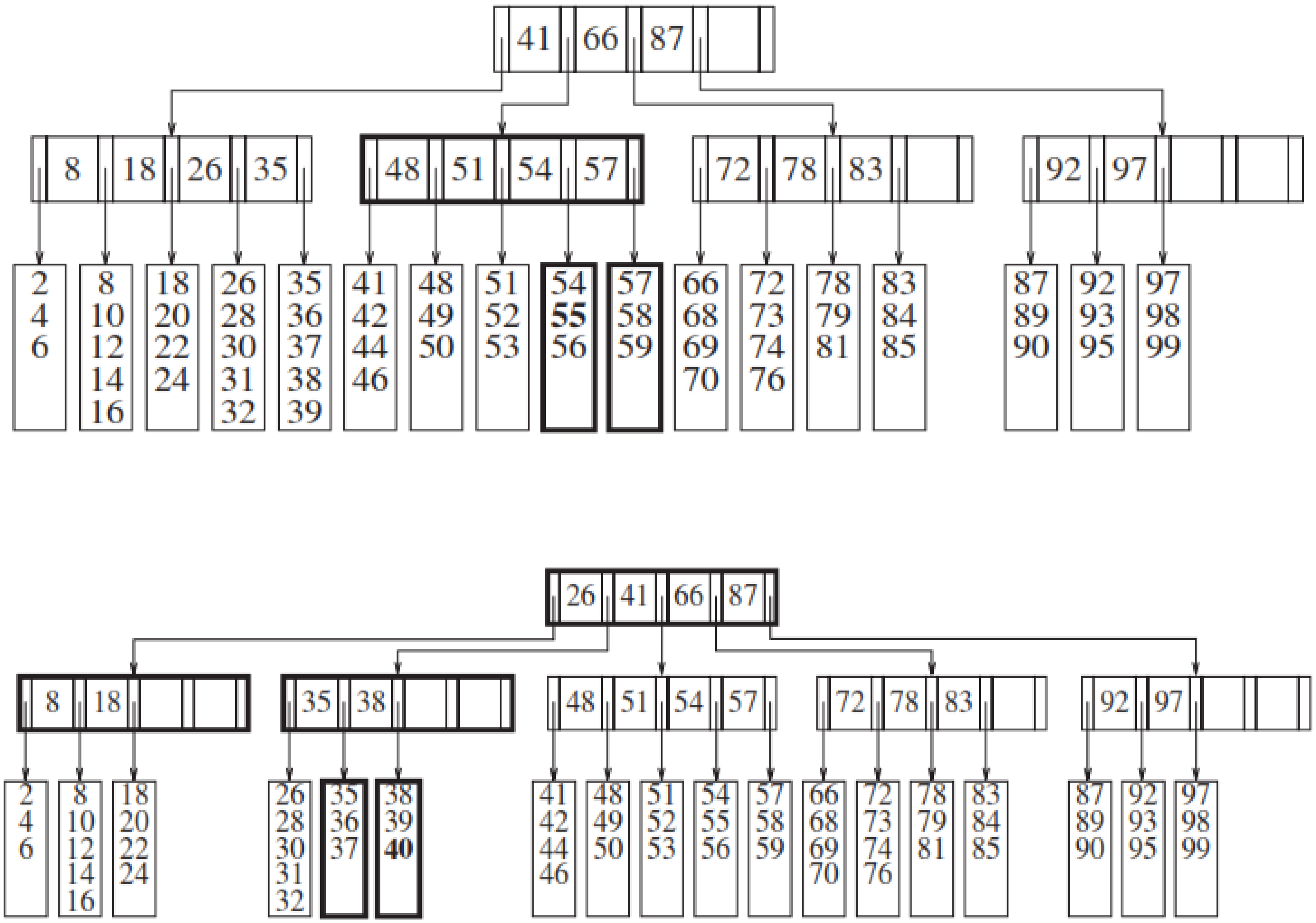


B-Trees

- insertion: splitting a parent – insert 40
  - what if the parent node already has all the child nodes it can possibly have?
    - split the parent node, and update its parent
    - repeat until we arrive at the root
    - if necessary, split the root into two nodes and create a new root with the two nodes as children
      - this is why the root is allowed as few as 2 children
      - thus, a B-tree grows at the root

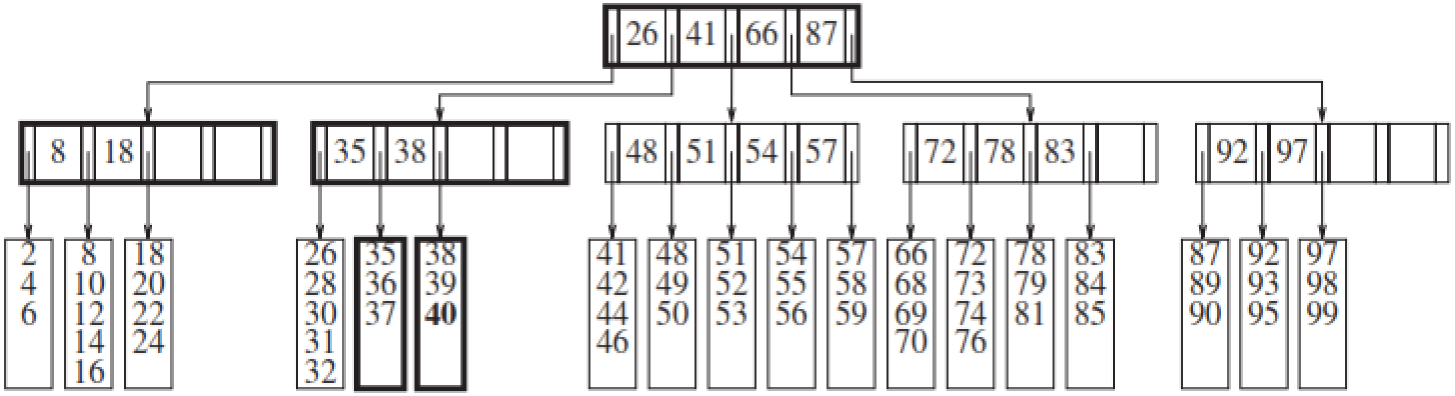
B-Trees

- insertion: splitting a parent – insert 40 (cont.)



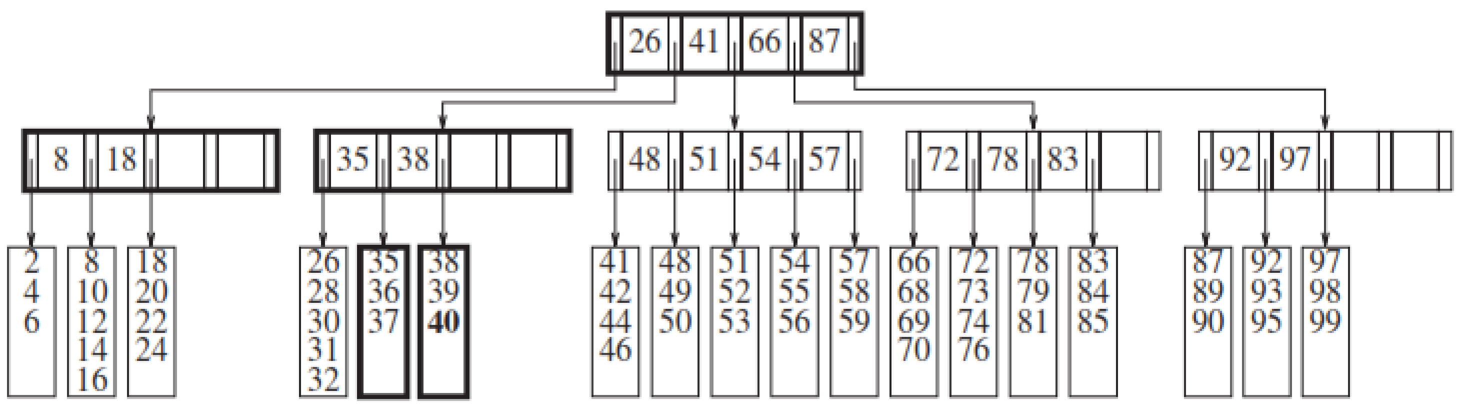
B-Trees

- insertion: other techniques – insert 29
  - put a child up for adoption if a neighbor has room
  - here, move 32 to the next leaf
  - modifies parent, but keeps nodes fuller and saves space in the long run



B-Trees

- deletion: delete 99
  - could bring leaf below <u>minimum</u> number of data items
    - adopt neighboring item if neighbor not at minimum
    - otherwise, <u>combine</u> with neighbor to form a full leaf
    - process could make its way up to the root
      - if root left with 1 child, remove root and make its child the new root of the tree



– deletion: delete 99 (cont.)