Chapter 10
Algorithm Design Techniques

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Dynamic Programming

- dynamic programming
  - very general approach to finding an optimal path through the state space of feasible states
  - state space may be represented as a directed graph with feasible states as nodes and feasible decisions as arcs
    - forms the decision network

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Dynamic Programming

- we apply dynamic programming to a problem when
  - the problem can be divided into stages with a decision made at each stage
  - each stage has a number of possible states associated with it
  - the decision made at each stage describes how the state at the current stage leads to the state at the next stage
  - given the current state, the optimal choice for the remaining stages does not depend on previous decisions or their associated states

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Dynamic Programming

- formulating dynamic programming recursions

The dynamic programming recursion (for minimization) is

$$f_t(i) = \min\{\text{cost during stage } t + f_{t+1}(\text{new state at stage } t+1)\}.$$

To derive the recursion we need to identify:

- the set of feasible decisions for the given state and stage;
- how the cost during the current stage t depends on t, the current state, and the decision chosen at stage t; and
- how the state at stage $t+1$ depends on t, the state at stage t, and the decision chosen at stage t.

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Dynamic Programming

- the knapsack problem

We have a knapsack that can hold a weight of at most W.

We can choose from T different types of articles to pack.

Each article of type t has (integer) weight w_t and (integer) value of v_t.

We wish to load a knapsack to maximize the total value of the articles included, subject to the capacity constraint.

Let

$$x_t = \text{number of articles of type } t \text{ included.}$$

An integer programming formulation is

$$\begin{aligned} \text{maximize} \quad & \textstyle\sum_t v_t x_t \\ \text{subject to} \quad & \textstyle\sum_t w_t x_t \le W \\ & x_t \ge 0 \text{ integer for all } t. \end{aligned}$$

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Dynamic Programming

- the knapsack problem (cont.)

The only resource here is weight.

Let $f_t(d)$ be the maximum value of items $t, t+1, \ldots, T$ if their combined weight is $\le d$.

The dynamic programming recursion is

$$\begin{aligned} f_{T+1}(d) &= 0; \\ f_t(d) &= \max_{x_t \in \mathbb{Z}_+} \{\, v_t x_t + f_{t+1}(d - w_t x_t) \mid w_t x_t \le d \,\}. \end{aligned}$$

The optimal solution we seek corresponds to $f_1(W)$.

We start by computing f_T for all possible states and work our way back to $f_1(W)$.

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Dynamic Programming

– the knapsack problem (cont.)

To illustrate this approach, suppose

item type	weight	value
1	4	11
2	3	7
3	5	12

and that $W = 10$.

Observe that we can work forwards from $t = 1$ and $d = 10$ to eliminate some states from consideration.

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Dynamic Programming

– the knapsack problem (cont.)

The preceding figure makes clear that we need only compute

$$f_3(0),\ f_3(1),\ f_3(2),\ f_3(3),\ f_3(4),\ f_3(6),\ f_3(7),\ f_3(10)$$

and

$$f_2(2),\ f_2(6),\ f_2(10),$$

and, of course, $f_1(10)$.

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Dynamic Programming

– the knapsack problem (cont.)

If our weight allowance d is 0 when we must decide how many of Item 3 to pack, then all we can do is choose 0 of the Item 3, for a net value of 0:

$$f_3(0) = 0.$$

Similarly,

$$f_3(1) = f_3(2) = f_3(3) = f_3(4) = 0.$$

because Item 3 weighs 5 pounds.

On the other hand,

$$f_3(5) = f_3(6) = f_3(7) = f_3(8) = f_3(9) = 12,$$

since in these cases we have 5–9 pounds of capacity available to us, so we can fit one of the 5 lb Item 3 into the knapsack.

Finally,

$$f_3(10) = 24,$$

since in this case we can space for 2 of Item 3.

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Dynamic Programming

– the knapsack problem (cont.)

In the next stage we compute

$$f_2(2),\ f_2(6),\ f_2(10).$$

We have

$$f_2(d) = \max_{x_2 \in \mathbb{Z}_+} \{\, 7x_2 + f_3(d - 3x_2) \mid 3x_2 \le d \,\}.$$

If $d = 2$ our only option is to choose none of Item 2, so

$$f_2(2) = \max_{x_2 \in \mathbb{Z}_+} \{\, 7x_2 + f_3(2 - 3x_2) \mid 3x_2 \le 2 \,\} = 0 + f_3(2) = 0.$$

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Dynamic Programming

– the knapsack problem (cont.)

On the other hand,

$$f_2(6) = \max_{x_2 \in \mathbb{Z}_+} \{\, 7x_2 + f_3(6 - 3x_2) \mid 3x_2 \le 6 \,\}$$
$$= \begin{cases} 0 + f_3(6) = 12 & x_2 = 0 \\ 7 + f_3(3) = 7 & x_2 = 1 \\ 14 + f_3(0) = 14 & x_2 = 2 \end{cases}$$

$$f_2(10) = \max_{x_2 \in \mathbb{Z}_+} \{\, 7x_2 + f_3(10 - 3x_2) \mid 3x_2 \le 10 \,\}$$
$$= \begin{cases} 0 + f_3(10) = 24 & x_2 = 0 \\ 7 + f_3(7) = 19 & x_2 = 1 \\ 14 + f_3(4) = 14 & x_2 = 2 \\ 21 + f_3(1) = 21 & x_2 = 3 \end{cases}$$

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Dynamic Programming

– the knapsack problem (cont.)

Finally,

$$f_1(10) = \max_{x_1 \in \mathbb{Z}_+} \{\, 11x_1 + f_2(10 - 4x_1) \mid 4x_1 \le 10 \,\}$$
$$= \begin{cases} 0 + f_2(10) = 24 & x_2 = 0 \\ 11 + f_2(6) = 25 & x_1 = 1 \\ 22 + f_2(2) = 22 & x_1 = 2 \end{cases}$$

The optimal strategy is

- one of Item 1,
- two of Item 2,
- zero of Item 3.

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Dynamic Programming

– an alternative approach

The following bottom-up approach leads to a simpler solution algorithm:

1. first determine how to fill a smaller knapsack optimally, then
2. use this knowledge to fill a larger knapsack optimally.

Let $V(w)$ denote the maximum value of a w-lb knapsack.

To fill a w-lb knapsack optimally, we must begin by packing an item. If we begin with an item of type t then the best value we can achieve is

$$v_t + \text{the best we can do with a } (w - w_t)\text{-lb knapsack.}$$

This leads to the recurrence

$$V(0) = 0$$
$$V(w) = \max_t \{\, v_t + V(w - w_t) \mid w_t \le w \,\}, \quad w > 0.$$

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Dynamic Programming

– an alternative approach (cont.)

To illustrate this approach, suppose

item type	weight	value
1	4	11
2	3	7
3	5	12

and that $W = 10$.

Clearly,

$$V(0) = V(1) = V(2) = 0,$$

since no item weighs 2 pounds or less, and

$$V(3) = 7$$

since only an item of type 2 will fit in the 3-lb knapsack.

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Dynamic Programming

– an alternative approach (cont.)

Now follow the recursion to fill out the values of V:

$$V(4) = \max \begin{cases} 11 + V(0) = 11 & \text{type 1} \\ 7 + V(1) = 7 & \text{type 2} \end{cases}$$

$$V(5) = \max \begin{cases} 11 + V(1) = 11 & \text{type 1} \\ 7 + V(2) = 7 & \text{type 2} \\ 12 + V(0) = 12 & \text{type 3} \end{cases}$$

$$V(6) = \max \begin{cases} 11 + V(2) = 11 & \text{type 1} \\ 7 + V(3) = 14 & \text{type 2} \\ 12 + V(1) = 12 & \text{type 3} \end{cases}$$

$$V(7) = \max \begin{cases} 11 + V(3) = 18 & \text{type 1} \\ 7 + V(4) = 18 & \text{type 2} \\ 12 + V(2) = 12 & \text{type 3} \end{cases}$$

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Dynamic Programming

– an alternative approach (cont.)

$$V(8) = \max \begin{cases} 11 + V(4) = 22 & \text{type 1} \\ 7 + V(5) = 19 & \text{type 2} \\ 12 + V(3) = 19 & \text{type 3} \end{cases}$$

$$V(9) = \max \begin{cases} 11 + V(5) = 23 & \text{type 1} \\ 7 + V(6) = 21 & \text{type 2} \\ 12 + V(4) = 23 & \text{type 3} \end{cases}$$

$$V(10) = \max \begin{cases} 11 + V(6) = 25 & \text{type 1} \\ 7 + V(7) = 25 & \text{type 2} \\ 12 + V(5) = 24 & \text{type 3} \end{cases}$$

Starting with a 10 lb knapsack, one optimal selection is given by

1. a type 1 item, leaving 10 - 4 = 6 lb;
2. a type 2 item, leaving 6 - 3 = 3 lb;
3. a type 2 item, leaving 3 - 3 = 0 lb.

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Dynamic Programming

– computational complexity

This DP approach requires we compute $V(0), \ldots, V(W)$, and each $V(w)$ requires we look at (at most) T sums.

Thus, $O(WT)$ operations are required.

However, the knapsack problem is NP-hard!

This DP solution of knapsack is a pseudo-polynomial time algorithm—the run-time is polynomial in the numeric value of the input W, not the number of bits in W (length of the input).

Suppose it takes $m > 1$ bits to represent W. This means $2^{m-1} \le W \le 2^m - 1$, so the DP approach is actually exponential in m.

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Multiplication

– how fast can we multiply?

If we multiply two n-digit numbers in the obvious way, the time required is proportional to n^2.

$$\begin{array}{rr} & 123 \\ \times & 456 \\ \hline & 738 \\ + & 7150 \\ + & 49200 \\ \hline & 56088 \end{array}$$

Can we do better?

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Multiplication

– standard multiplication

Given two $2n$-bit numbers $u = (u_{2n-1} \cdots u_1 u_0)_2$ and $v = (v_{2n-1} \cdots v_1 v_0)_2$, we can write

$$u = 2^n U_1 + U_0$$
$$v = 2^n V_1 + V_0,$$

where $U_1 = (u_{2n-1} \cdots u_1 u_n)_2$ consists of the n most significant bits of u, while $U_0 = (u_{n-1} \cdots u_1 u_0)_2$ are the n least significant bits, and similarly for V_1, V_0.

The obvious way of multiplication is

$$uv = (2^{2n} + 2^n)U_1V_1 + 2^n(U_1V_0 + U_0V_1) + (2^n + 1)U_0V_0.$$

Multiplications by powers of 2 are $O(n)$ left shifts and $\pm$ is also $O(n)$.

Recursion for runtime T:

$$T(2n) = 4T(n) + cn \quad \Rightarrow \quad T(n) = \Theta(n^2).$$

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Multiplication

– multiplication by divide and conquer

A faster approach (A. A. Karatsuba (1962)):

$$uv = (2^{2n} + 2^n)U_1V_1 + 2^n(U_1 - U_0)(V_1 - V_0) + (2^n + 1)U_0V_0.$$

This is true since

$$(U_1 - U_0)(V_1 - V_0) = U_1V_1 - U_1V_0 - U_0V_1 + U_0V_0.$$

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Multiplication

– recursion for the complexity

Let $T(2n)$ be the time needed to compute the product of two $2n$-bit numbers via

$$uv = (2^{2n} + 2^n)U_1V_1 + 2^n(U_1 - U_0)(V_1 - V_0) + (2^n + 1)U_0V_0.$$

How many multiplications are on the right?

There are only 3 multiplications, since the multiplications by powers of 2 are just shifts. The cost of the shifts are $\propto n$

There are also a bunch of additions, but this work is also $\propto n$.

This leads to the recursion

$$T(2n) = 3T(n) + cn$$
$$T(1) = c'.$$

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Multiplication

– solution via reduction

Suppose $2n = 2^m$. From the recursion

$$T(2n) = 3T(n) + cn$$
$$T(1) = c'$$

we obtain the following:

$$T(n) = 3T(n/2) + c(n/2),$$

so

$$\begin{aligned} T(2n) &= 3(3T(n/2) + c(n/2)) + cn \\ &= 9T(n/2) + 3c(n/2)) + cn \end{aligned}$$

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Multiplication

– solution via reduction (cont.)

Again applying the recursion, we obtain

$$T(n/2) = 3T(n/4) + c(n/4),$$

so

$$T(2n) = 27T(n/4) + 9c(n/4) + 3c(n/2) + cn.$$

Now a pattern has emerged: we conjecture that after k steps of this process,

$$T(2n) = 3^k T(2^{-k}2n) + \sum_{i=0}^{k-1} \left(\frac{3}{2}\right)^i cn.$$

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Multiplication

– complexity

If we repeat this $k = m$ times, so $2^k = 2^m = 2n$, we obtain

$$\begin{aligned} T(2n) &= 3^m T(1) + \sum_{i=0}^{m-1} \left(\frac{3}{2}\right)^i cn \\ &= 3^m c' + cn\frac{1 - (3/2)^m}{1 - (3/2)} = 3^m c' + 2cn((3/2)^m - 1) \end{aligned}$$

The dominant term is $3^m c'$, and

$$3^m = 3^{\lg n} = (2^{\lg 3})^{\lg n} = (2^{\lg n})^{\lg 3} = n^{\lg 3} = n^{1.5850\ldots}$$

so the divide-and-conquer algorithm is $\Theta(n^{1.585})$.

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Matrix Multiplication

Suppose A and B are $n \times n$ matrices.

How fast can we compute AB?

Standard matrix multiplication:

```
C = 0 // C <- A*B
for i = 1 to n {
  for j = 1 to n {
    for k = 1 to n {
      C(i,j) += A(i,k)*B(i,k)
    }
  }
}
```

This is $\Theta(n^3)$.

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Matrix Multiplication

– block matrix multiplication

Suppose $n = 2^m$ for some m. Write A and B in terms of $n/2 \times n/2$ blocks:

$$\begin{pmatrix} C_{11} & C_{12} \\ C_{21} & C_{22} \end{pmatrix} = \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix} \begin{pmatrix} B_{11} & B_{12} \\ B_{21} & B_{22} \end{pmatrix}$$

Standard matrix multiplication:

$$\begin{aligned} C_{11} &= A_{11}B_{11} + A_{12}B_{21} \\ C_{12} &= A_{11}B_{12} + A_{12}B_{22} \\ C_{21} &= A_{21}B_{11} + A_{22}B_{21} \\ C_{22} &= A_{21}B_{12} + A_{22}B_{22} \end{aligned}$$

8 $n/2 \times n/2$ matrix products and 4 $n/2 \times n/2$ matrix additions.

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Matrix Multiplication

– Strassen's fast matrix multiplication (1969)

$$\begin{aligned} \text{I} &= (A_{11} + A_{22})(B_{11} + B_{22}) \\ \text{II} &= (A_{21} + A_{22})B_{11} \\ \text{III} &= A_{11}(B_{12} - B_{22}) \\ \text{IV} &= A_{22}(-B_{11} + B_{21}) \\ \text{V} &= (A_{11} + A_{12})B_{22} \\ \text{VI} &= (-A_{11} + A_{21})(B_{11} + B_{22}) \\ \text{VII} &= (A_{12} - A_{22})(B_{21} + B_{22}) \end{aligned}$$

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Matrix Multiplication

– Strassen's fast matrix multiplication (1969) (cont.)

$$\begin{aligned} C_{11} &= \text{I} + \text{IV} - \text{V} + \text{VII} \\ C_{12} &= \text{III} + \text{V} \\ C_{21} &= \text{II} + \text{IV} \\ C_{22} &= \text{I} + \text{III} - \text{II} + \text{VI} \end{aligned}$$

7 $n/2 \times n/2$ matrix products and 18 $n/2 \times n/2$ matrix additions.

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Matrix Multiplication

– Strassen's trick

Strassen trades an $O((n/2)^3)$ matrix product for 14 $O((n/2)^2)$ matrix additions.

Now apply the algorithm recursively to compute the $n/2 \times n/2$ matrix products.

If $T(n)$ is the time it takes to compute an $n \times n$ matrix product using Strassen, then

$$T(n) = 7\,T(n/2) + 18n^2.$$

This recurrence leads to

$$T(n) = O(n^{\log_2 7}) \approx O(n^{2.81}) \quad (4.7\,n^{2.81}).$$

This is better than the $O(n^3)$ complexity of standard matrix multiplication!

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Matrix Multiplication

– Strassen's – practical concerns

This discussion assumes A and B are square but there exist variants for rectangular matrices.

We can compute and use the terms I–VII one at a time, so we need not store all of them.

Some extra storage is needed because of the recursion.

At some point in the recursion standard matrix multiplication becomes more efficient so we switch.

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Matrix Multiplication

– Strassen's algorithm for matrix inversion

Strassen's algorithm for inversion has a complexity bounded by $5.64\, n^{\log_2 7}$.

Let

$$A = \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix}, \qquad A^{-1} = \begin{pmatrix} C_{11} & C_{12} \\ C_{21} & C_{22} \end{pmatrix}.$$

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Matrix Multiplication

– Strassen's algorithm for matrix inversion (cont.)

Then

$$\begin{aligned} \mathsf{I} &= A_{11}^{-1} \\ \mathsf{II} &= A_{21}\mathsf{I} \\ \mathsf{III} &= \mathsf{I}A_{12} \\ \mathsf{IV} &= A_{21}\mathsf{III} \\ \mathsf{V} &= \mathsf{IV} - A_{22} \\ \mathsf{VI} &= \mathsf{V}^{-1} \\ C_{12} &= \mathsf{III} \cdot \mathsf{VI} \\ C_{21} &= \mathsf{VI} \cdot \mathsf{II} \\ \mathsf{VII} &= \mathsf{III} \cdot C_{21} \\ C_{11} &= \mathsf{I} - \mathsf{VII} \\ C_{22} &= -\mathsf{VI}. \end{aligned}$$

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Matrix Multiplication

– state of the art

Winograd (1972): Variant of Strassen with 7 matrix-matrix products and 15 matrix-matrix additions $\Rightarrow \Theta(n^{\log_2 7})$ with better constant.

Pan (1978): $\Theta(n^{2.795})$.

Coppersmith and Winograd (1990): $\Theta(n^{2.376})$.

Le Gall's variant of Coppersmith and Winograd (2014): $\Theta(n^{2.373})$—best known.

Cohn, Kleinberg, Szegedy, Umans (2005): Conjectures based on group theory which, if true, implies $\Theta(n^{2+\varepsilon})$ for any $\varepsilon > 0$.

Clearly $\Theta(n^2)$ is a lower bound on matrix multiplication—it takes n^2 operations just to write down the answer.

Conjecture: Matrix multiplication can be performed in $\Theta(n^{2+\varepsilon})$ for any $\varepsilon > 0$.

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