Bits, Bytes, and Integers

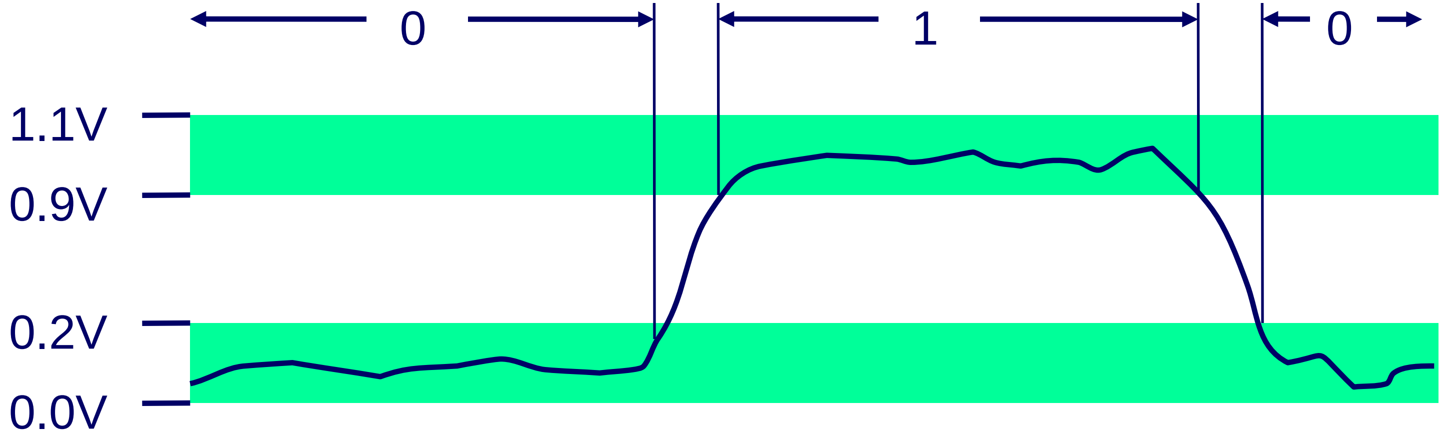
with contributions from Dr. Bin Ren, College of William & Mary

Bits, Bytes, and Integers

- **Representing information as bits**
- **Bit-level manipulations**
- **Integers**
  - Representation: unsigned and signed
  - Conversion, casting
  - Expanding, truncating
  - Addition, negation, multiplication, shifting
  - Summary
- **Representations in memory, pointers, strings**

Everything is Bits

- **Each bit is 0 or 1**
- **By encoding/interpreting sets of bits in various ways**
  - Computers determine what to do (instructions)
  - … and represent and manipulate numbers, sets, strings, etc…
- **Why bits? Electronic Implementation**
  - Easy to store with bi-stable elements
  - Reliably transmitted on noisy and inaccurate wires



The Decimal System and Bases

- **base 10 (decimal): digits 0-9**
  - e.g., $316_{10} = 3 \times 10^2 + 1 \times 10^1 + 6 \times 10^0 = 300 + 10 + 6$

- **in the decimal system, 10 is the base, or radix**

- **any integer > 1 can be a base**

- **base 2 has two digits: 0 and 1**
  - bit = **bi**nary dig**it**
  - $1011_2 = 1 \times 2^3 + 0 \times 2^2 + 1 \times 2^1 + 1 \times 2^0 =$

Converting to Decimal

- **Base eight (octal): digits 0-7**
  - $474_8 \;=$
- **Base 16 (hexadecimal): digits 0-9 and A-F**
  - $13C_{16} \;=$
- **Base 2 (binary): digits 0, 1**
  - $100110_2 \;=$
- **In general, radix *r* representations use the first *r* chars in {0…9, A…Z} and have the form $d_{n-1}d_{n-2}\ldots d_1d_0$.**
  - Summing $d_{n-1} \times r^{n-1} + d_{n-2} \times r^{n-2} + \ldots + d_0 \times r^0$ converts to base 10.

Converting from Decimal to Binary

- **convert 1693 to binary**
- **use a divisor of 2 to obtain the following sequence of quotients and remainders**

dividend	quotient	remainder
1693	846	1
846	423	0
423	211	1
211	105	1
105	52	1
52	26	0
26	13	0
13	6	1
6	3	0
3	1	1
1	0	1

- read remainders in reverse order $1693_{10} = 11010011101_2$

More Base Conversion Practice

- **convert to base 10 by multiplication of powers**
  - $10012_5 = (\qquad\qquad)_{10}$
- **convert from base 10 by repeated division**
  - $632_{10} = (\qquad\quad)_8$
- **converting base *x* to base *y*: convert base *x* to base 10 then convert base 10 to base *y***

More Base Conversion Practice

- **Convert from base 10**
  - $123_{10} = (\qquad\qquad)_3$ and check
  - $1234_{10} = (\qquad\qquad)_{16}$ and check
- **Another way to convert from decimal to base n**

	n^8	n^7	n^6	n^5	n^4	n^3	n^2	n^1	n^0
for n = 2	256	128	64	32	16	8	4	2	1

  - From LEFT TO RIGHT, ask "how many" and subtract
  - $(219)_{10} = (\qquad\qquad)_2 = (\qquad\qquad)_{16}$

Converting Between Hex and Binary

- **chart of values**

decimal	hex	binary	decimal	hex	binary
0	0	0000	8	8	1000
1	1	0001	9	9	1001
2	2	0010	10	A	1010
3	3	0011	11	B	1011
4	4	0100	12	C	1100
5	5	0101	13	D	1101
6	6	0110	14	E	1110
7	7	0111	15	F	1111

- **to convert from binary to hex**
  - start at right of binary number
  - convert each group of 4 digits into a hex value
  - e.g., convert 11011101100_2 to hex

binary:	0110	1110	1100
hex:	6	E	C

Converting Between Hex and Binary

- **chart of values**

decimal	hex	binary	decimal	hex	binary
0	0	0000	8	8	1000
1	1	0001	9	9	1001
2	2	0010	10	A	1010
3	3	0011	11	B	1011
4	4	0100	12	C	1100
5	5	0101	13	D	1101
6	6	0110	14	E	1110
7	7	0111	15	F	1111

- **to convert from hex to binary**
  - replace each hex digit with its binary equivalent
  - e.g., convert $8A5_{16}$ to binary

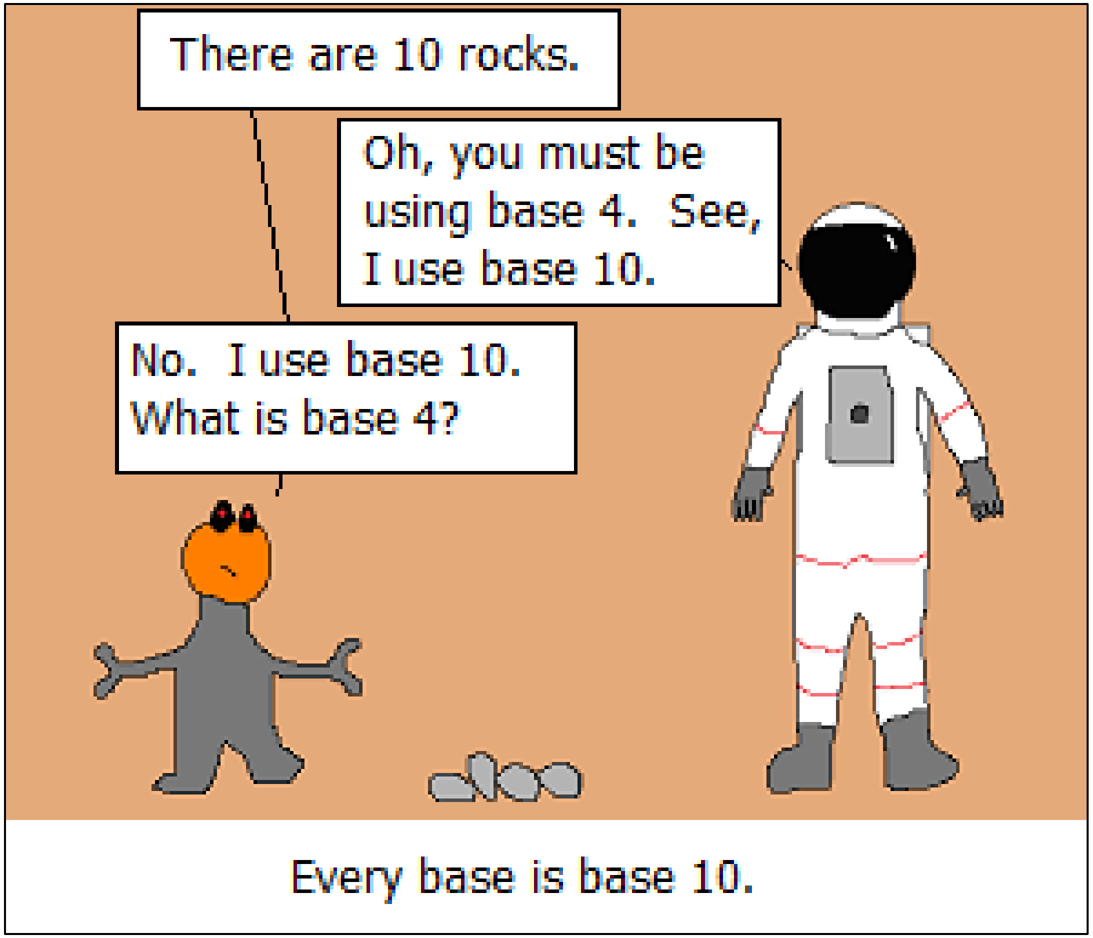
hex:	8	A	5
binary:	1000	1010	0101

Octal

- **$2^4 = 16$ and $2^3 = 8$**
  - power = # of bits per hex/octal digit
- **Binary to Hex**
  - every 4 bits = 1 hex digit
- **Octal – base 8**
  - digits 0-7
- **Binary to Octal**
  - Every 3 bits = 1 octal digit

DEC	OCT	HEX	BIN	Notes
	0	0	0	-
1	1	1	1	2^0
2	2	2	10	2^1
3	3	3	11	
4	4	4	100	2^2
5	5	5	101	
6	6	6	110	
7	7	7	111	
8	10	8	1000	2^3
9	11	9	1001	
10	12	A	1010	
11	13	B	1011	
12	14	C	1100	
13	15	D	1101	
14	16	E	1110	
15	17	F	1111	

Every Base is Base 10



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In general, $10_X = X_{10}$

- $10_2 = 2$
- $10_3 = 3$
- $10_4 = 4$
- $10_5 = 5$
- $10_6 = 6$
- $10_7 = 7$
- $10_8 = 8$
- $10_9 = 9$
- $10_{10} = 10$

Other Binary Numbers

- **Base 2 Number Representation**
  - Represent 15213_{10} as 11101101101101_2
  - Represent 1.20_{10} as $1.0011001100110011[0011]\ldots_2$
  - Represent 1.5213×10^4 as $1.1101101101101_2 \times 2^{13}$

Encoding Byte Values

- **Byte = 8 bits**
  - Binary 00000000_2 to 11111111_2
  - Decimal: 0_{10} to 255_{10}
  - Hexadecimal 00_{16} to FF_{16}
    - useful for writing binary values concisely
    - write $FA1D37B_{16}$ in C as
      - 0xFA1D37B
      - 0xfa1d37b

Hex	Decimal	Binary
0	0	0000
1	1	0001
2	2	0010
3	3	0011
4	4	0100
5	5	0101
6	6	0110
7	7	0111
8	8	1000
9	9	1001
A	10	1010
B	11	1011
C	12	1100
D	13	1101
E	14	1110
F	15	1111

Example Data Representations

C Data Type	Typical 32-bit	Typical 64-bit	x86-64
`char`	1	1	1
`short`	2	2	2
`int`	4	4	4
`long`	4	8	8
`float`	4	4	4
`double`	8	8	8
`long double`	–	–	10/16
pointer	4	8	8

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- **Bit-level manipulations**
- Integers
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Boolean Algebra

- **Developed by George Boole in 19th Century**
  - Algebraic representation of logic
    - Encode "True" as 1 and "False" as 0

And

- **A&B = 1 when both A=1 and B=1**

&	0	1
0	0	0
1	0	1

Or

- **A|B = 1 when either A=1 or B=1**

|	0	1
0	0	1
1	1	1

Not

- **~A = 1 when A=0**

~	
0	1
1	0

Exclusive-Or (Xor)

- **A^B = 1 when either A=1 or B=1, but not both**

^	0	1
0	0	1
1	1	0

General Boolean Algebras

- **Operate on Bit Vectors**
  - Operations applied bitwise

```
  01101001     01101001     01101001
& 01010101   | 01010101   ^ 01010101   ~ 01010101
  01000001     01111101     00111100     10101010
```

- **All of the Properties of Boolean Algebra Apply**

Example: Representing & Manipulating Sets

- **Representation**
  - Width w bit vector represents subsets of {0, …, w–1}
  - $a_j = 1$ if $j \in A$

    - 01101001 { 0, 3, 5, 6 }
    - *76543210*

    - 01010101 { 0, 2, 4, 6 }
    - *76543210*
- **Operations**

&	Intersection	01000001	{ 0, 6 }
|	Union	01111101	{ 0, 2, 3, 4, 5, 6 }
^	Symmetric difference	00111100	{ 2, 3, 4, 5 }
~	Complement	10101010	{ 1, 3, 5, 7 }

Bit-Level Operations in C

- **Operations &, |, ~, ^ available in C**
  - Apply to any “integral” data type
    - long, int, short, char, unsigned
  - View arguments as bit vectors
  - Arguments applied bit-wise
- **Examples (char data type)**
  - ~0x41 → 0xBE
    - $\sim 01000001_2 \rightarrow 10111110_2$
  - ~0x00 → 0xFF
    - $\sim 00000000_2 \rightarrow 11111111_2$
  - 0x69 & 0x55 → 0x41
    - 01101001_2 & $01010101_2 \rightarrow 01000001_2$
  - 0x69 | 0x55 → 0x7D
    - $01101001_2 \mid 01010101_2 \rightarrow 01111101_2$

Contrast: Logic Operations in C

- **Contrast to Bit-level Operators**
  - &&, ||, !
    - View 0 as "false"
    - Anything nonzero as "true"
    - Always return 0 or 1
    - Early termination
- **Examples (char data type)**
  - !0x41 → 0x00
  - !0x00 → 0x01
  - !!0x41 → 0x01

  - 0x69 && 0x55 → 0x01
  - 0x69 || 0x55 → 0x01
  - p && *p (avoids null pointer access)

Contrast: Logic Operations in C

- **Contrast to Bit-level Operators**
  - &&, ||, !
    - View 0 as “Fal
    - Anything nonze
    - Alway
    - Early
- **Example**
  - !0x41 →
  - !0x00 →
  - !!0x41 →
  - 0x69 && 0x55 → 0x01
  - 0x69 || 0x55 → 0x01
  - p && *p (avoids null pointer access)

Watch out for && vs. & (and || vs. |)… one of the more common oopsies in C programming

Shift Operations

- **Left Shift: `x << y`**
  - Shift bit-vector **`x`** left **`y`** positions
    - – Throw away extra bits on left
    - Fill with 0's on right
- **Right Shift: `x >> y`**
  - Shift bit-vector **`x`** right **`y`** positions
    - Throw away extra bits on right
  - Logical shift
    - Fill with 0's on left
  - Arithmetic shift
    - Replicate most significant bit on left
- **Undefined Behavior**
  - Shift amount < 0 or ≥ word size

Argument `x`	01100010
<< 3	00010*000*
Log. >> 2	*00*011000
Arith. >> 2	*00*011000

Argument `x`	10100010
<< 3	00010*000*
Log. >> 2	*00*101000
Arith. >> 2	*11*101000

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Finite Precision

- **the space to store integers in a computer is limited**
  - forced to deal with finite precision

- **different machines use a varying number of bits for its word size, from 4 to 256 bits**
  - nominal size of integer and pointer data
  - 32 and 64 bits are the current preferred sizes

- **in general, we can store 2^n different values with n bits**
  - 1 bit: 2 values (0 and 1)
  - 2 bits: 4 values (00, 01, 10, 11)
  - 4 bits: 16 values
    - we've seen 0..15, but no negative values

Number of Values

- **Address space depends on word size → $2^{\text{word-size-in-\#bits}}$**
  - Is it big enough?
    - 64-bit high-end machines becoming more prevalent
    - Portability issues – insensitive to sizes of different data types

# bytes	# bits	# of values ($2^{\#bits}$)	low	high
1	8	256		
2	16	65536		
3	24	16777216		
4	32	4294967296		
5	40	1.09951E+12		
6	48	2.81475E+14		
7	56	7.20576E+16		
8	64	1.84467E+19		
9	72	4.72237E+21		
10	80	1.20893E+24		
11	88	3.09485E+26		
12	96	7.92282E+28		
13	104	2.02824E+31		
14	112	5.1923E+33		
15	120	1.32923E+36		
16	128	3.40282E+38		

Negative Values

- **so far, we've seen the number of positive integers possible, but no negative values**

- **common sense tells us to split the number of bit patterns into two groups of roughly the same size: one for positive values and one for negative values**
  - don't forget 0

- **many ways to split these values have been developed over the years**
  - two's complement is the most popular
  - unsigned represents only non-negative values (positive values and 0)

Encoding Integers

Unsigned

$$B2U(X) = \sum_{i=0}^{w-1} x_i \cdot 2^i$$

Two's Complement

$$B2T(X) = -x_{w-1} \cdot 2^{w-1} + \sum_{i=0}^{w-2} x_i \cdot 2^i$$

Sign Bit

```
short int x =  15213;
short int y = -15213;
```

- **C `short` 2 bytes long**

	Decimal	Hex	Binary
`x`	15213	`3B 6D`	`00111011 01101101`
`y`	-15213	`C4 93`	`11000100 10010011`

- **Sign Bit**
  - For 2's complement, most significant bit indicates sign
    - 0 for nonnegative
    - 1 for negative

Two-complement Encoding Example (Cont.)

```
x =      15213: 00111011 01101101
y =     -15213: 11000100 10010011
```

Weight	15213		-15213	
1	1	1	1	1
2	0	0	1	2
4	1	4	0	0
8	1	8	0	0
16	0	0	1	16
32	1	32	0	0
64	1	64	0	0
128	0	0	1	128
256	1	256	0	0
512	1	512	0	0
1024	0	0	1	1024
2048	1	2048	0	0
4096	1	4096	0	0
8192	1	8192	0	0
16384	0	0	1	16384
-32768	0	0	1	-32768
Sum		**15213**		**-15213**

Two's Complement – Simple Conversion

- **conversion from positive to negative using a two-step process**
  - reverse the bits of the positive representation
  - add 1 to the result
  - e.g.,

00001001	9
11110110	reverse all bits
11110111	add 1 = -9

- **only one representation for 0**

  00000000

  11111111 + 1 = 00000000

- **one more negative number than positive number**

Two’s Complement – Alternate Conversion

- **alternate conversion using a two-step process**
  - reading from right to left, copy all values up to and including the first 1
  - reverse the remainder of the bits
  - e.g.,

    00011100 28

    11100100 -28

- **positive numbers do not need conversion**

Numeric Ranges

- **Unsigned Values**
  - *UMin* = 0
    000…0
  - *UMax* = $2^w - 1$
    111…1

- **Two's Complement Values**
  - *TMin* = -2^{w-1}
    100…0
  - *TMax* = $2^{w-1} - 1$
    011…1
- **Other Values**
  - Minus 1
    111…1

Values for *W* = 16

	Decimal	Hex	Binary
UMax	65535	FF FF	11111111 11111111
TMax	32767	7F FF	01111111 11111111
TMin	-32768	80 00	10000000 00000000
-1	-1	FF FF	11111111 11111111
0	0	00 00	00000000 00000000

Values for Different Word Sizes

	W			
	8	**16**	**32**	**64**
UMax	255	65,535	4,294,967,295	18,446,744,073,709,551,615
TMax	127	32,767	2,147,483,647	9,223,372,036,854,775,807
TMin	-128	-32,768	-2,147,483,648	-9,223,372,036,854,775,808

- **Observations**
  - $|TMin| \quad = \quad TMax + 1$
    - Asymmetric range
  - $UMax \quad = \quad 2 * TMax + 1$

- **C Programming**
  - #include <limits.h>
  - Declares constants, e.g.,
    - ULONG_MAX
    - LONG_MAX
    - LONG_MIN
  - Values are platform-specific

Unsigned & Signed Numeric Values

X	**B2U(*X*)**	**B2T(*X*)**
0000	0	0
0001	1	1
0010	2	2
0011	3	3
0100	4	4
0101	5	5
0110	6	6
0111	7	7
1000	8	–8
1001	9	–7
1010	10	–6
1011	11	–5
1100	12	–4
1101	13	–3
1110	14	–2
1111	15	–1

- **Equivalence**
  - Same encodings for nonnegative values
- **Uniqueness**
  - Every bit pattern represents unique integer value
  - Each representable integer has unique bit encoding
- **$\Rightarrow$ Can Invert Mappings**
  - U2B(x) = B2U^{-1}(x)
    - Bit pattern for unsigned integer
  - T2B(x) = B2T^{-1}(x)
    - Bit pattern for two's comp integer

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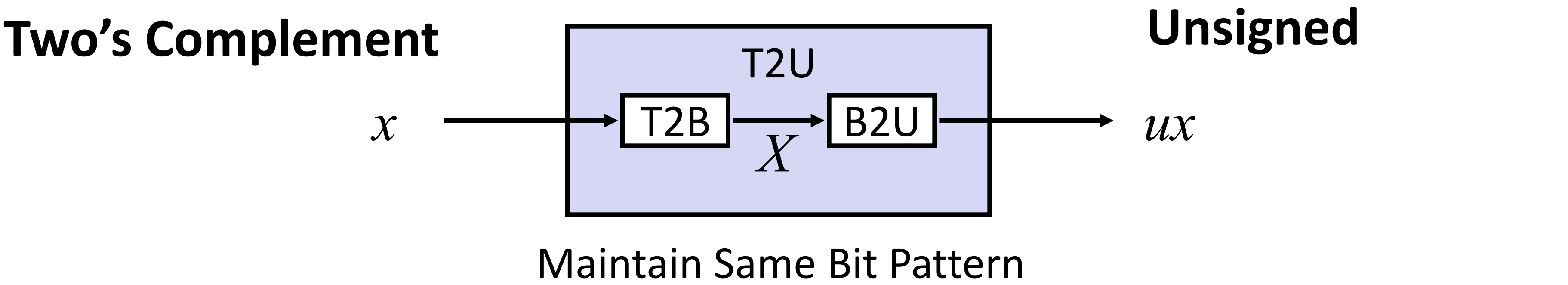
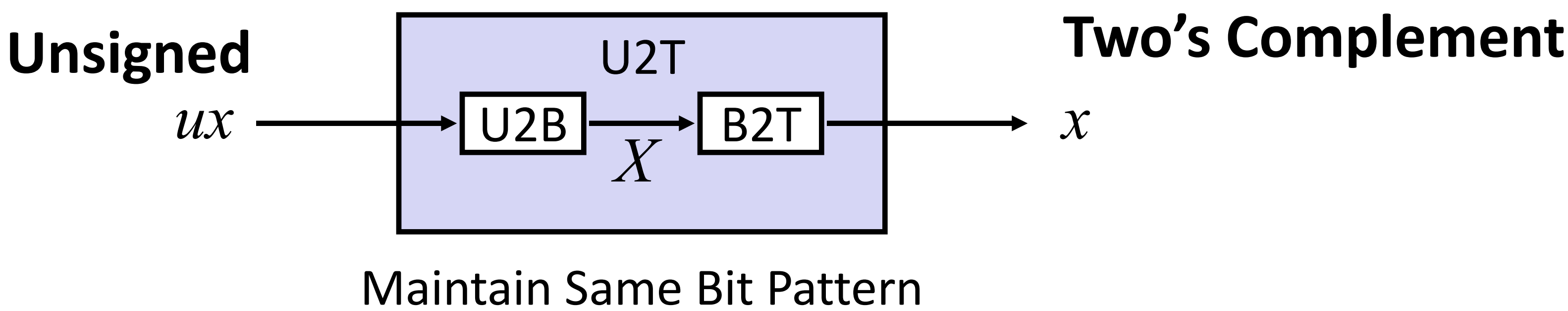
Mapping Between Signed & Unsigned





- **Mappings between unsigned and two's complement numbers:**
  Keep bit representations and reinterpret

Mapping Signed ↔ Unsigned

Bits	Signed	Unsigned
0000	0	0
0001	1	1
0010	2	2
0011	3	3
0100	4	4
0101	5	5
0110	6	6
0111	7	7
1000	-8	8
1001	-7	9
1010	-6	10
1011	-5	11
1100	-4	12
1101	-3	13
1110	-2	14
1111	-1	15

→ T2U →

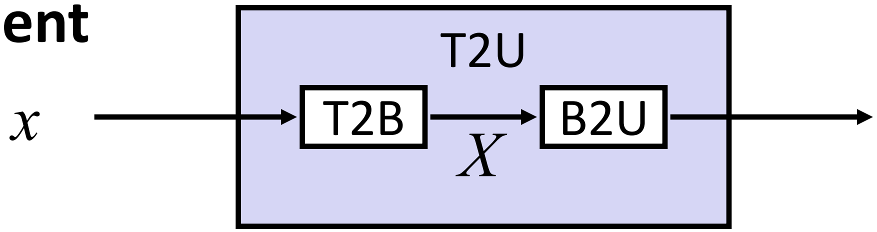
← U2T ←

Mapping Signed ↔ Unsigned

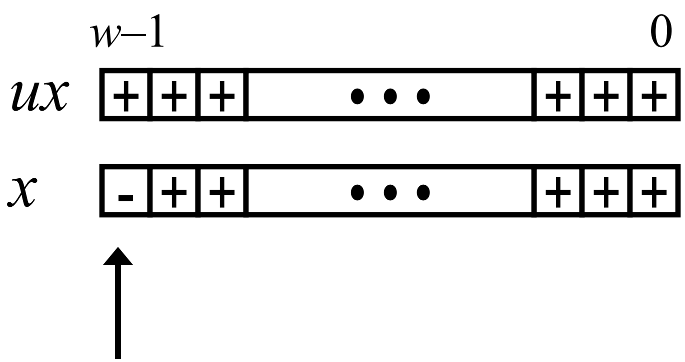
Bits	Signed		Unsigned
0000	0		0
0001	1		1
0010	2		2
0011	3	=	3
0100	4		4
0101	5		5
0110	6		6
0111	7		7
1000	-8		8
1001	-7		9
1010	-6		10
1011	-5	+/- 16	11
1100	-4		12
1101	-3		13
1110	-2		14
1111	-1		15

Relation between Signed & Unsigned

Two's Complement **Unsigned**



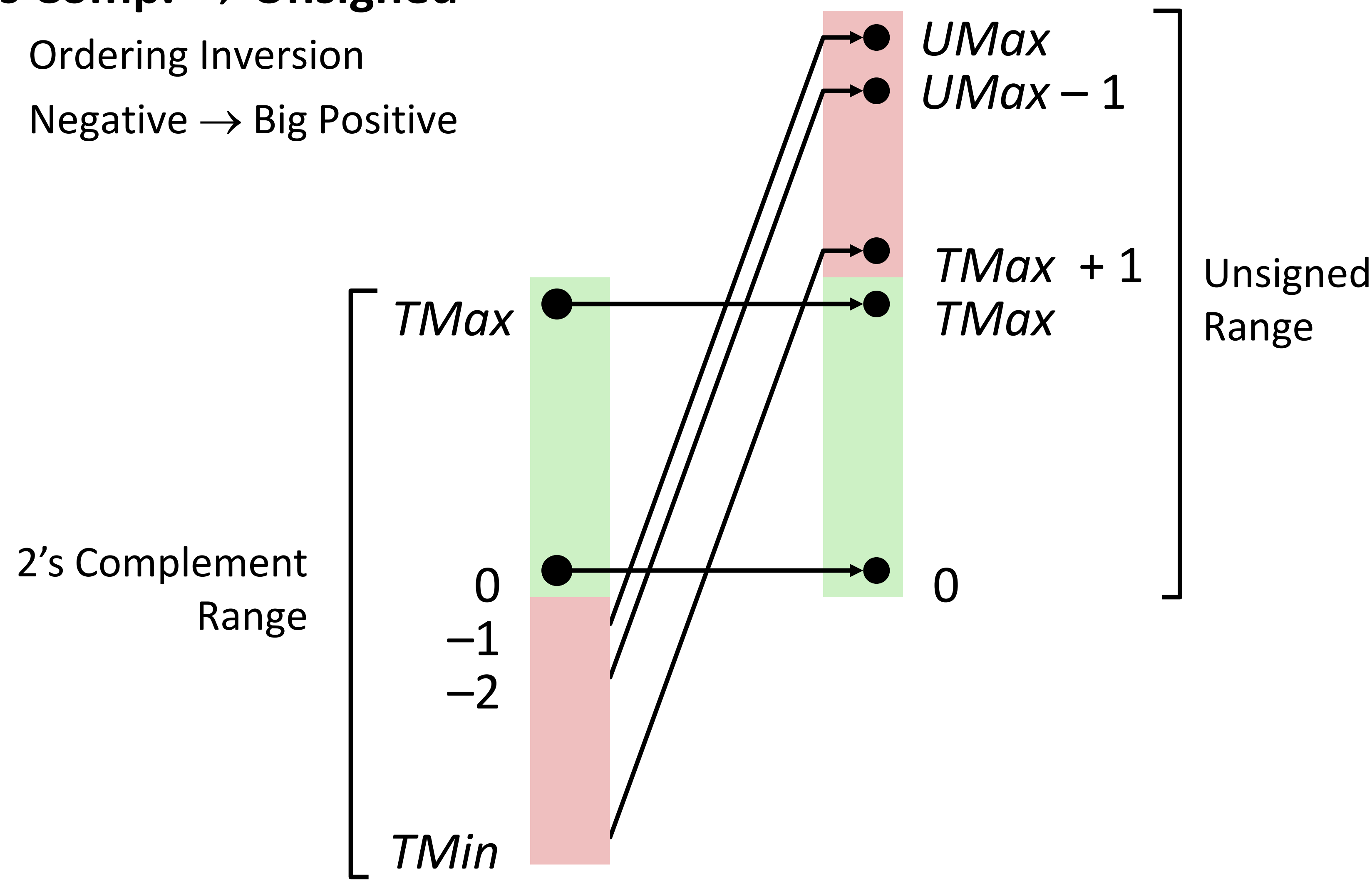
Maintain Same Bit Pattern



Large negative weight
becomes
Large positive weight

Conversion Visualized

- **2's Comp. → Unsigned**
  - Ordering Inversion
  - Negative → Big Positive



Signed vs. Unsigned in C

- **Constants**
  - By default are considered to be signed integers
  - Unsigned if have “U” as suffix

    `0U, 4294967259U`
- **Casting**
  - Explicit casting between signed & unsigned same as U2T and T2U

```
int tx, ty;
unsigned ux, uy;
tx = (int) ux;
uy = (unsigned) ty;
```

  - Implicit casting also occurs via assignments and procedure calls

```
tx = ux;
uy = ty;
```

Casting Surprises

- **Expression Evaluation**
  - If there is a mix of unsigned and signed in single expression, ***signed values implicitly cast to unsigned***
  - Including comparison operations **<, >, ==, <=, >=**
  - Examples for *W* = 32: **TMIN = -2,147,483,648 , TMAX = 2,147,483,647**

Constant$_1$	Constant$_2$	Relation	Evaluation
0	0U	**==**	**unsigned**
-1	0	**<**	**signed**
-1	0U	**>**	**unsigned**
2147483647	-2147483647-1	**>**	**signed**
2147483647U	-2147483647-1	**<**	**unsigned**
-1	-2	**>**	**signed**
(unsigned)-1	-2	**>**	**unsigned**
2147483647	2147483648U	**<**	**unsigned**
2147483647	(int) 2147483648U	**>**	**signed**

Summary
Casting Signed ↔ Unsigned: Basic Rules

- **Bit pattern is maintained**
- **But reinterpreted**
- **Can have unexpected effects: adding or subtracting 2^w**

- **Expression containing signed and unsigned int**
  - `int` is cast to `unsigned`!!

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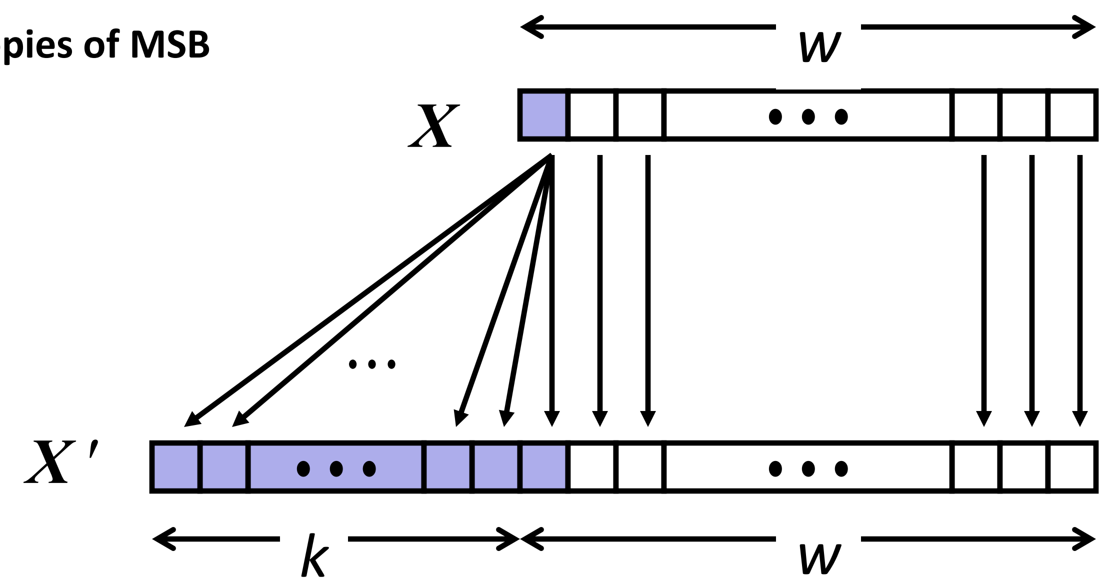
Sign Extension

- **Task:**
  - Given w-bit signed integer x
  - Convert it to $w+k$-bit integer with same value
- **Rule:**
  - Make k copies of sign bit:
  - $X' = \underbrace{x_{w-1},\ldots, x_{w-1}}_{k \textbf{ copies of MSB}}, x_{w-1}, x_{w-2},\ldots, x_0$



Sign Extension Example

```
short int x =  15213;
int       ix = (int) x;
short int y = -15213;
int       iy = (int) y;
```

	Decimal	Hex	Binary
x	15213	3B 6D	00111011 01101101
ix	15213	00 00 3B 6D	00000000 00000000 00111011 01101101
y	-15213	C4 93	11000100 10010011
iy	-15213	FF FF C4 93	11111111 11111111 11000100 10010011

- Converting from smaller to larger integer data type
- C automatically performs sign extension

Summary: Expanding, Truncating: Basic Rules

- **Expanding (e.g., short int to int)**
  - Unsigned: zeros added
  - Signed: sign extension
  - Both yield expected result

- **Truncating (e.g., unsigned to unsigned short)**
  - Unsigned/signed: bits are truncated
  - Result reinterpreted
  - Unsigned: mod operation
  - Signed: similar to mod
  - For small numbers yields expected behavior

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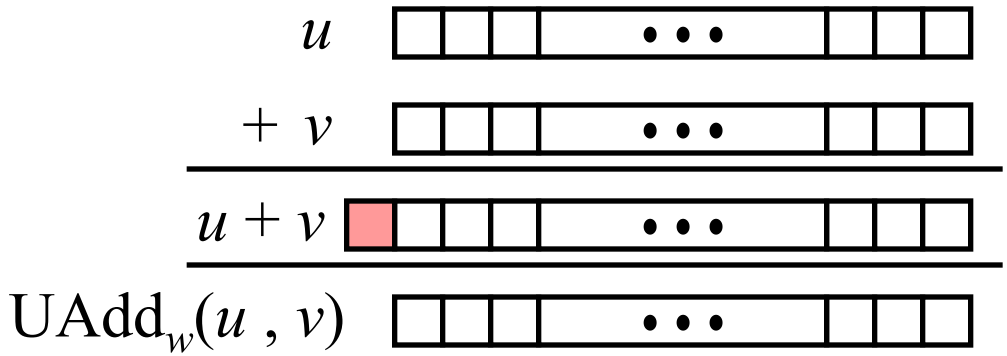
Unsigned Addition

Operands: *w* bits

True Sum: *w*+1 bits

Discard Carry: *w* bits



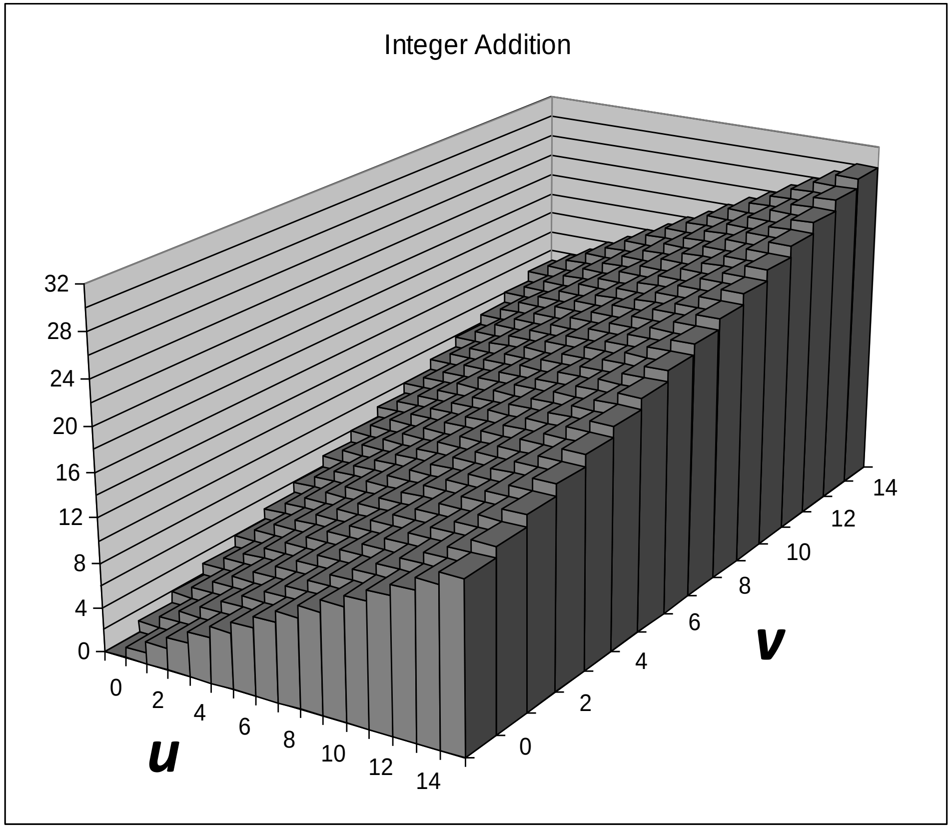
- **Standard Addition Function**
  - Ignores carry output
- **Implements Modular Arithmetic**

$s \quad = \quad \mathrm{UAdd}_w(u \, , v) \quad = \quad u + v \mod 2^w$

Visualizing (Mathematical) Integer Addition

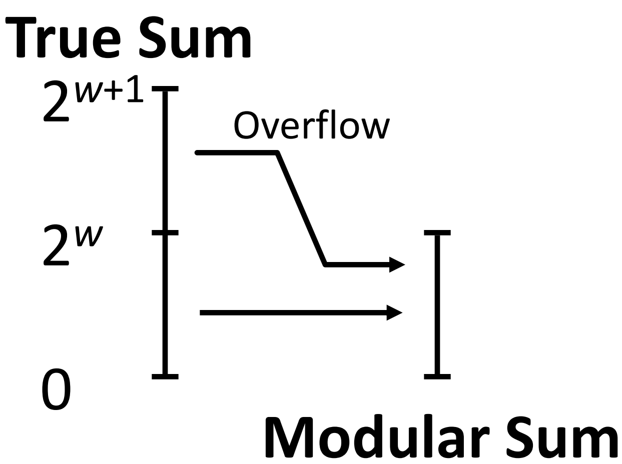
- **Integer Addition**
  - 4-bit integers *u*, *v*
  - Compute true sum $\text{Add}_4(u, v)$
  - Values increase linearly with *u* and *v*
  - Forms planar surface

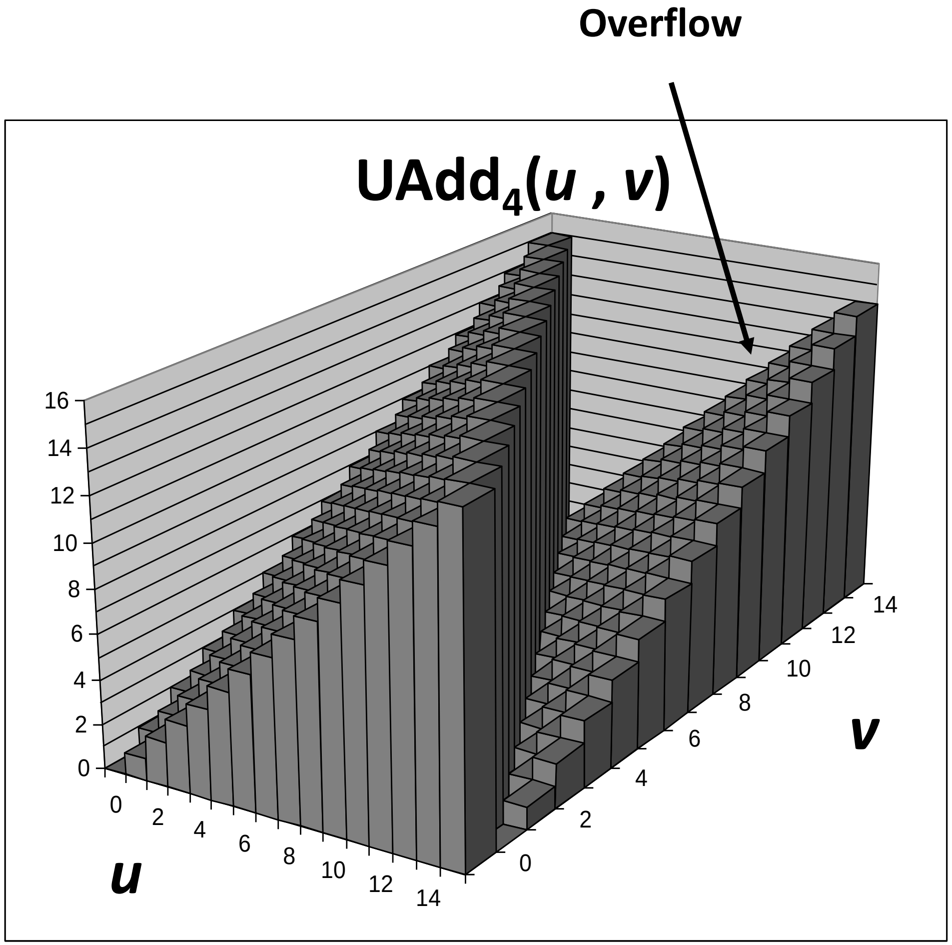
$\text{Add}_4(u, v)$



Visualizing Unsigned Addition

- **Wraps Around**
  - If true sum ≥ 2^w
  - At most once



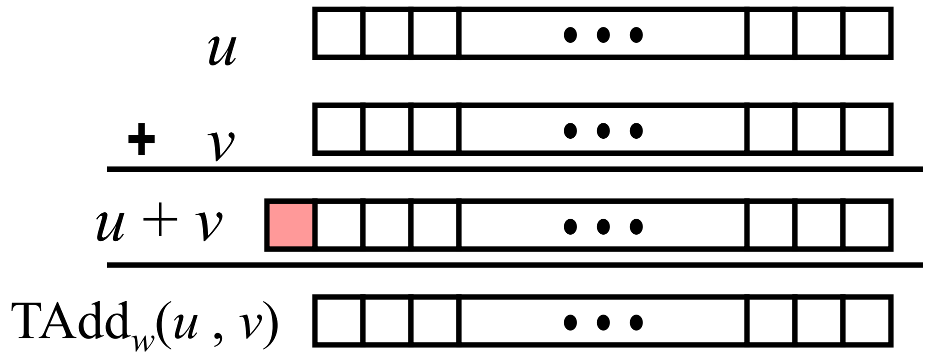


Two's Complement Addition

Operands: *w* bits

True Sum: *w*+1 bits

Discard Carry: *w* bits

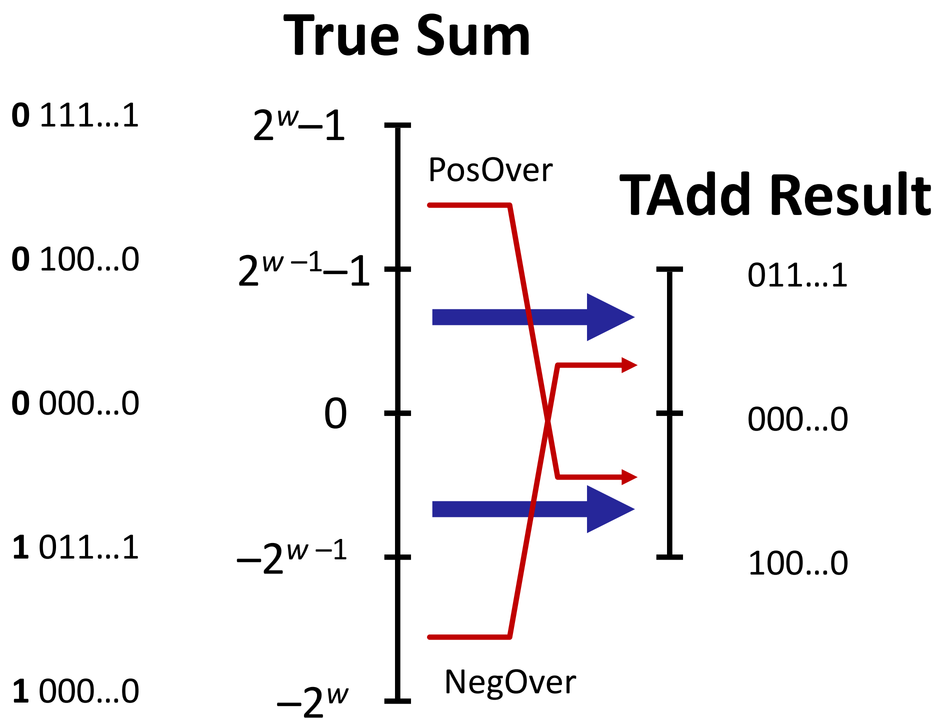


- **TAdd and UAdd have Identical Bit-Level Behavior**
  - Signed vs. unsigned addition in C:

    ```
    int s, t, u, v;
    s = (int) ((unsigned) u + (unsigned) v);
    t = u + v
    ```
  - Will give `s == t`

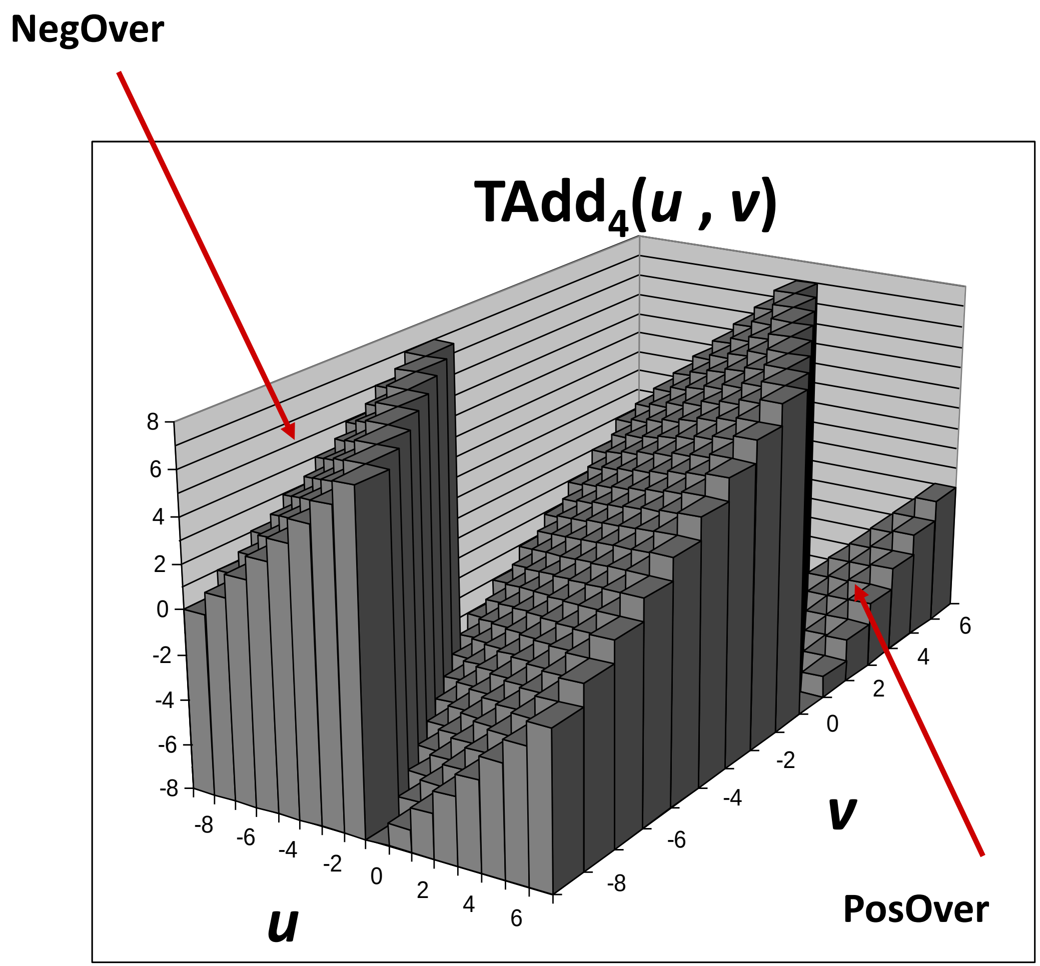
TAdd Overflow

- **Functionality**
  - True sum requires w+1 bits
  - Drop off MSB
  - Treat remaining bits as 2's comp. integer



Visualizing 2's Complement Addition

- **Values**
  - 4-bit two's comp.
  - Range from -8 to +7
- **Wraps Around**
  - If sum $\geq 2^{w-1}$
    - Becomes negative
    - At most once
  - If sum $< -2^{w-1}$
    - Becomes positive
    - At most once



Binary Addition

- **Goal: compute sum of *w*-bit numbers *x*, *y***
  - Either signed or unsigned
- **Binary Addition Basics**

$$
\begin{array}{r} 0 \\ +\,0 \\ \hline 0 \end{array}
\qquad
\begin{array}{r} 1 \\ +\,0 \\ \hline 1 \end{array}
\qquad
\begin{array}{r} 0 \\ +\,1 \\ \hline 1 \end{array}
\qquad
\begin{array}{r} 1 \\ +\,1 \\ \hline 10 \end{array}
$$

Binary Addition

- **Examples**

Problem in base ten	Problem in two's complement	Answer in base ten
3 + 2	0011 + 0010 = 0101	5
−3 + −2	1101 + 1110 = 1011	−5
7 + −5	0111 + 1011 = 0010	2

Binary Addition

- Using 6 bits we can represent values from -32 to 31, so what happens when we try to add 19 plus 14 or -19 and -14

19 +14 33	010011 +001110 100001	we have added two positive numbers and gotten a negative result – this is positive overflow
-19 -14 -33	101101 +110010 011111	we have added two negative numbers and gotten a positive result – this is negative overflow

Binary Addition

- 8-bit binary addition

```
  0101 0101
+ 0001 1001
  0110 1110
```
no left bit discarded
no overflow

```
  0111 0101
+ 0101 1011
  1101 0000
```
no left bit discarded
positive overflow

```
  0111 0101
+ 1101 1011
1 0101 0000
```
left bit discarded
no overflow

```
  1111 0101
+ 1101 1011
1 1101 0000
```
left bit discarded
no overflow

```
  1111 0101
+ 1000 0011
1 0111 1000
```
left bit discarded
negative overflow

Multiplication

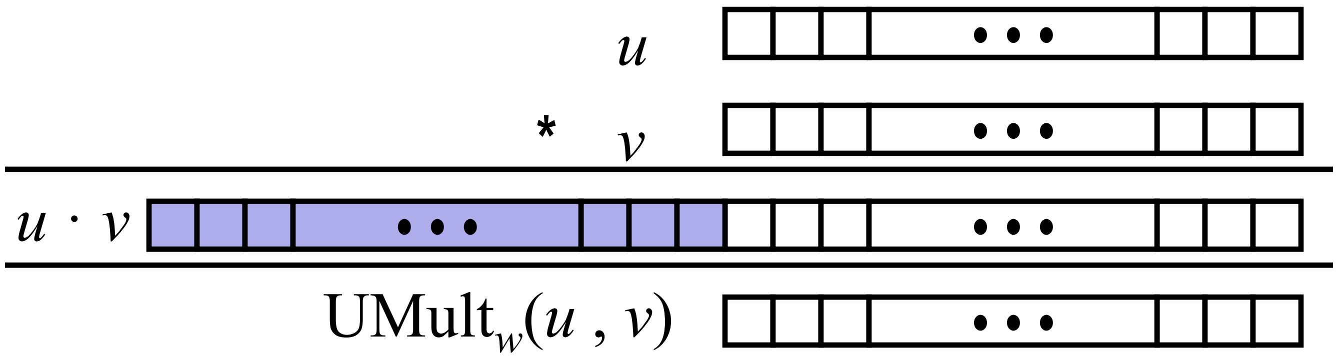
- **Goal: Computing Product of *w*-bit numbers *x*, *y***
  - Either signed or unsigned
- **But, exact results can be bigger than *w* bits**
  - Unsigned: up to 2*w* bits
    - Result range: $0 \le x * y \le (2^{w} - 1)^{2} = 2^{2w} - 2^{w+1} + 1$
  - Two's complement min (negative): Up to 2*w*-1 bits
    - Result range: $x * y \ge (-2^{w-1}) * (2^{w-1} - 1) = -2^{2w-2} + 2^{w-1}$
  - Two's complement max (positive): Up to 2*w* bits, but only for $(TMin_{w})^{2}$
    - Result range: $x * y \le (-2^{w-1})^{2} = 2^{2w-2}$
- **So, maintaining exact results…**
  - would need to keep expanding word size with each product computed
  - is done in software, if needed
    - e.g., by "arbitrary precision" arithmetic packages

Unsigned Multiplication in C

Operands: *w* bits

True Product: 2**w* bits

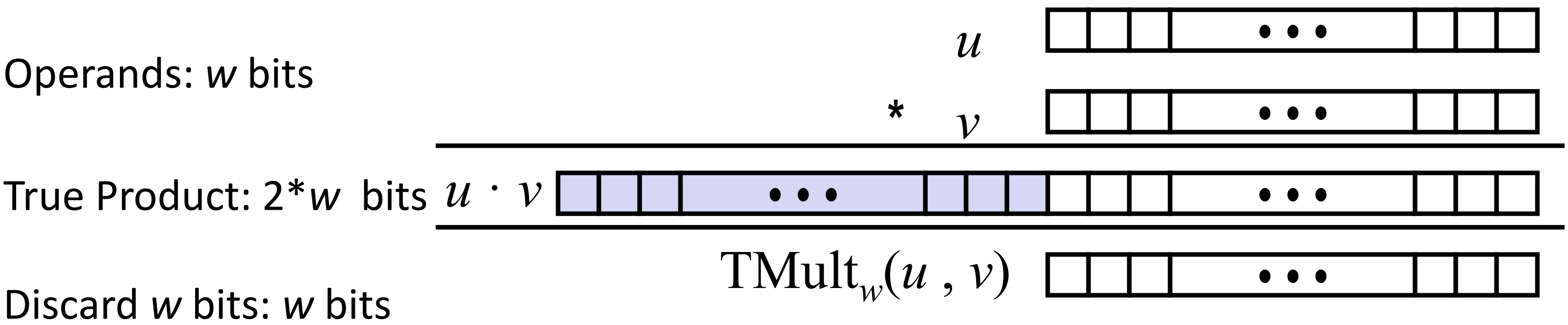
Discard *w* bits: *w* bits



- **Standard Multiplication Function**
  - Ignores high order *w* bits
- **Implements Modular Arithmetic**

$\mathrm{UMult}_w(u, v) = u \cdot v \bmod 2^w$

Signed Multiplication in C



- **Standard Multiplication Function**
  - Ignores high order *w* bits
  - Some of which are different for signed vs. unsigned multiplication
  - Lower bits are the same

Binary Multiplication

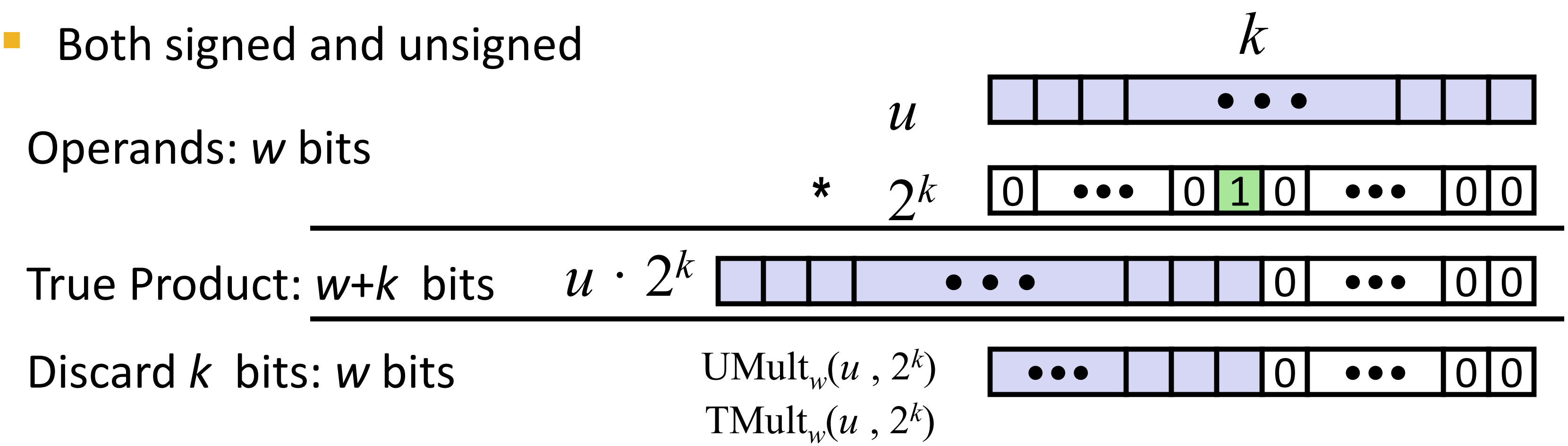
- **8-bit binary multiplication**

```
      01010101
    x 00011001
      --------
      01010101
   01010101
  01010101
  ------------
  100001001101
```

truncated: 0100 1101

Power-of-2 Multiply with Shift

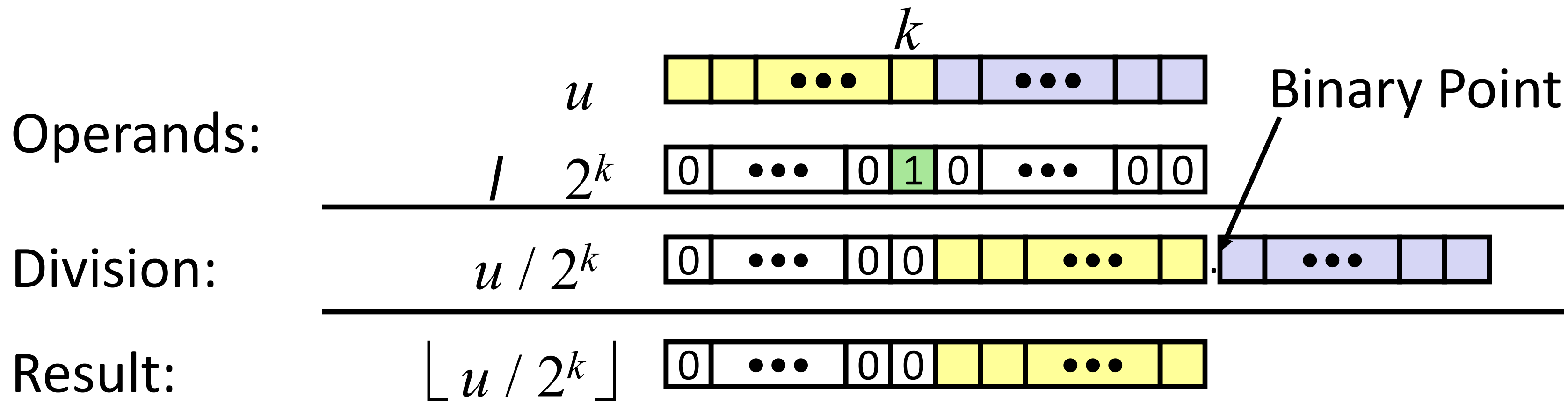
- **Operation**
  - **`u << k`** gives **`u * 2`**^{k}
  - Both signed and unsigned



- **Examples**
  - **`u << 3 == u * 8`**
  - **`(u << 5) - (u << 3) == u * 24`**
  - Most machines shift and add faster than multiply
    - Compiler generates this code automatically

Unsigned Power-of-2 Divide with Shift

- **Quotient of Unsigned by Power of 2**
  - `u >> k` gives $\lfloor$ `u / 2`^{k} $\rfloor$
  - Uses logical shift



	Division	Computed	Hex	Binary
`x`	15213	15213	`3B 6D`	`00111011 01101101`
`x >> 1`	7606.5	7606	`1D B6`	`00011101 10110110`
`x >> 4`	950.8125	950	`03 B6`	`00000011 10110110`
`x >> 8`	59.4257813	59	`00 3B`	`00000000 00111011`

Shifting and Overflow

- **since an arithmetic left shift is the same as multiplying by 2, we may run out of space, resulting in overflow**
  - ex., 8-bit unsigned: 0010 1110 << 3 = 0111 0000 (46 * 8 = 368, not 112)
  - ex., 8-bit signed: 0010 1110 << 2 = 1011 1000 (46 * 4 = 184, not -72)
  - ex., 8-bit signed: 1110 1110 << 3 = 0111 0000 (-18 * 8 = -144, not 112)
  - ex., 8-bit signed: 1110 1110 << 2 = 1011 1000 (-18 * 4 = -72 OK)

- **overflow limitations**
  - not valid with logical shifts
  - not possible using right shifts
  - determined by 1 bits shifting off left
    - if 1 bits used for sign extension, no overflow unless sign change
  - can also occur by 0 bits shifting off left
    - change sign of result

Bits, Bytes, and Integers

- Representing information as bits
- Bit-level manipulations
- **Integers**
  - Representation: unsigned and signed
  - Conversion, casting
  - Expanding, truncating
  - Addition, negation, multiplication, shifting
  - **Summary**
- Representations in memory, pointers, strings

Arithmetic: Basic Rules

- **Addition:**
  - Unsigned/signed: Normal addition followed by truncate, same operation on bit level
  - Unsigned: addition mod 2^w
    - Mathematical addition + possible subtraction of 2^w
  - Signed: modified addition mod 2^w (result in proper range)
    - Mathematical addition + possible addition or subtraction of 2^w

- **Multiplication:**
  - Unsigned/signed: Normal multiplication followed by truncate, same operation on bit level
  - Unsigned: multiplication mod 2^w
  - Signed: modified multiplication mod 2^w (result in proper range)

Why Should I Use Unsigned?

- ***Don't*** **use without understanding implications**
  - Easy to make mistakes

```
unsigned i;
for (i = cnt-2; i >= 0; i--)
  a[i] += a[i+1];
```

  - Can be very subtle

```
#define DELTA sizeof(int)
int i;
for (i = CNT; i-DELTA >= 0; i-= DELTA)
  . . .
```

Counting Down with Unsigned

- **Proper way to use unsigned as loop index**

```
unsigned i;
for (i = cnt-2; i < cnt; i--)
  a[i] += a[i+1];
```

- **See Robert Seacord, *Secure Coding in C and C++***
  - C Standard guarantees that unsigned addition will behave like modular arithmetic
    - 0 – 1 → *UMax*
- **Even better**

```
size_t i;
for (i = cnt-2; i < cnt; i--)
  a[i] += a[i+1];
```

  - Data type **`size_t`** defined as unsigned value with length = word size
  - Code will work even if **`cnt`** = *UMax*
  - What if **`cnt`** is signed and < 0?

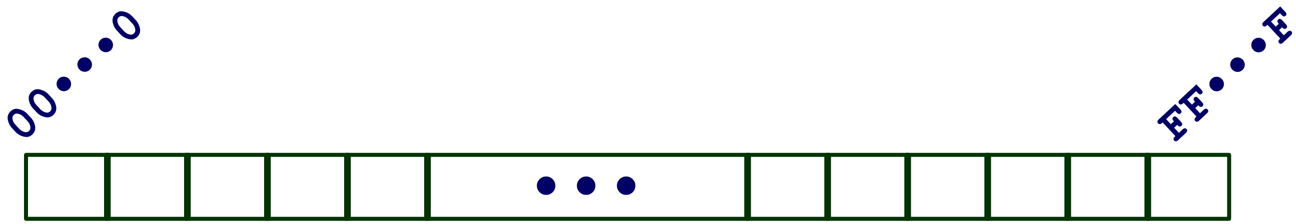
Why Should I Use Unsigned? (cont.)

- ***Do*** **Use When Performing Modular Arithmetic**
  - Multiprecision arithmetic
- ***Do*** **Use When Using Bits to Represent Sets**
  - Logical right shift, no sign extension

Bits, Bytes, and Integers

- Representing information as bits
- Bit-level manipulations
- Integers
  - Representation: unsigned and signed
  - Conversion, casting
  - Expanding, truncating
  - Addition, negation, multiplication, shifting
  - Summary
- **Representations in memory, pointers, strings**

Byte-Oriented Memory Organization



- **Programs refer to data by address**
  - Conceptually, envision it as a very large array of bytes
    - In reality, it's not, but can think of it that way
  - An address is like an index into that array
    - and, a pointer variable stores an address

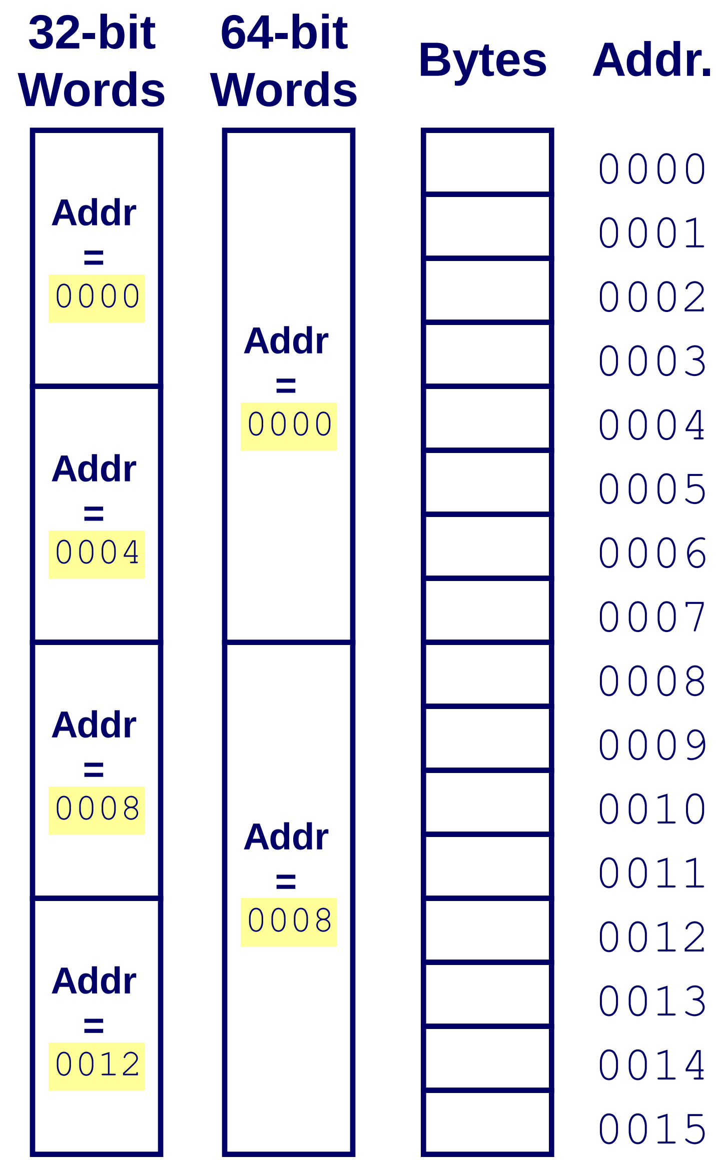
- **Note: system provides private address spaces to each "process"**
  - Think of a process as a program being executed
  - So, a program can clobber its own data, but not that of others

Machine Words

- **Any given computer has a "Word Size"**
  - Nominal size of integer-valued data
    - and of addresses
  - Until recently, most machines used 32 bits (4 bytes) as word size
    - Limits addresses to 4GB (2^{32} bytes)
  - Increasingly, machines have 64-bit word size
    - Potentially, could have 18 EB (exabytes) of addressable memory
    - That's 18.4 X 10^{18}
  - Machines still support multiple data formats
    - Fractions or multiples of word size
    - Always integral number of bytes

Word-Oriented Memory Organization

- **Addresses Specify Byte Locations**
  - Address of first byte in word
  - Addresses of successive words differ by 4 (32-bit) or 8 (64-bit)



Example Data Representations

C Data Type	Typical 32-bit	Typical 64-bit	x86-64
`char`	1	1	1
`short`	2	2	2
`int`	4	4	4
`long`	4	8	8
`float`	4	4	4
`double`	8	8	8
`long double`	−	−	10/16
pointer	4	8	8

Byte Ordering

- **So, how are the bytes within a multi-byte word ordered in memory?**
- **Conventions**
  - Big Endian: Sun, PPC Mac, Internet
    - Least significant byte has highest address
  - Little Endian: x86, ARM processors running Android, iOS, and Windows
    - Least significant byte has lowest address

Byte Ordering Example

- **Example**
  - Variable x has 4-byte value of 0x01234567
  - Address given by &x is 0x100

Big Endian

		0x100	0x101	0x102	0x103		
		01	23	45	67		

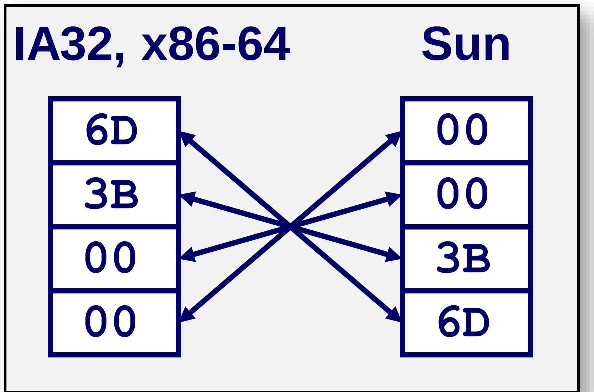
Little Endian

		0x100	0x101	0x102	0x103		
		67	45	23	01		

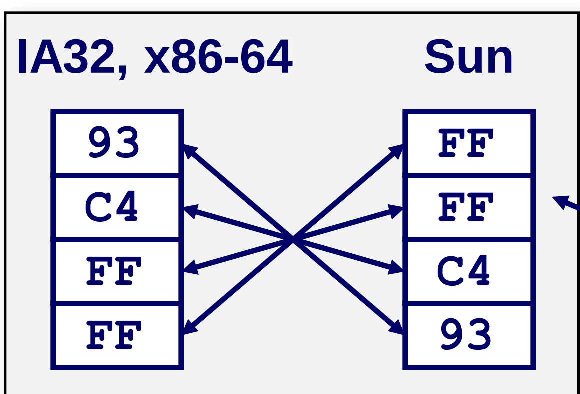
Representing Integers

Decimal:	15213			
Binary:	0011	1011	0110	1101
Hex:	3	B	6	D

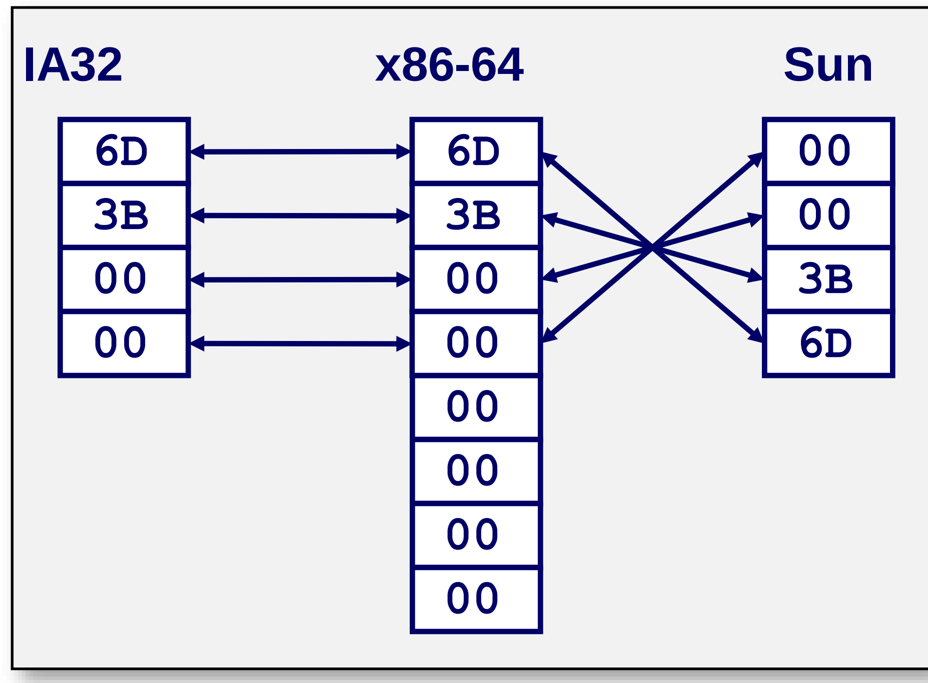
```
int A = 15213;
```




```
int B = -15213;
```




```
long int C = 15213;
```



Two's complement representation

Examining Data Representations

- **Code to Print Byte Representation of Data**
  - Casting pointer to unsigned char * allows treatment as a byte array

```
typedef unsigned char *pointer;

void show_bytes(pointer start, size_t len){
  size_t i;
  for (i = 0; i < len; i++)
    printf(”%p\t0x%.2x\n",start+i, start[i]);
  printf("\n");
}
```

Printf directives:
%p**:** Print pointer
%x**:** Print Hexadecimal

`show_bytes` Execution Example

```
int a = 15213;
printf("int a = 15213;\n");
show_bytes((pointer) &a, sizeof(int));
```

Result (Linux x86-64):

```
int a = 15213;
0x7fffb7f71dbc    6d
0x7fffb7f71dbd    3b
0x7fffb7f71dbe    00
0x7fffb7f71dbf    00
```

Representing Pointers

```
int B = -15213;
int *P = &B;
```

Sun	IA32	x86-64
EF	AC	3C
FF	28	1B
FB	F5	FE
2C	FF	82
		FD
		7F
		00
		00

Different compilers & machines assign different locations to objects

Even get different results each time run program

Representing Strings

```
char S[6] = "18213";
```

- **Strings in C**
  - Represented by array of characters
  - Each character encoded in ASCII format
    - Standard 7-bit encoding of character set
    - Character “0” has code 0x30
      - Digit *i* has code 0x30+*i*
  - String should be null-terminated
    - Final character = 0
- **Compatibility**
  - Byte ordering not an issue