5.3 Permutations and combinations

Reading: Rosen 5.3 and 5.5

- **Permutation:** A permutation of a set of distinct elements is an ordered arrangement of these elements. An r-permutation is an ordered arrangement of r elements of a set. The number of r-permutations of a set of n elements is denoted by $P(n,r)$. For positive integers n and r with $1 \leq r \leq n$, we have $P(n,r) = n(n-1)(n-2)\cdots(n-r+1) = \frac{n!}{(n-r)!}$.

  Example 1: How many ways can we select three students from a group five to stand in line for a picture? $P(5,3) = 5 \cdot 4 \cdot 3 = 60$.

  Example 2: How many permutations of the letters $ABCDEFGH$ contain the string ABC? We can treat ABC as one letter, then we get $6! = 720$.

- **Combination:** An r-combination of elements of a set is an unordered selection of r elements from the set, which is simply a subset of size r of the set. the number of r-combinations of a set with n distinct elements is denoted by $C(n,r)$, also $\binom{n}{r}$. For any nonnegative integers n and r with $0 \leq r \leq n$, we have $C(n,r) = \frac{n!}{r!(n-r)!} = C(n,n-r)$.

  Example 3: How many committees of three students can be formed from a group of five students? $C(5,3) = 10$.

  Example 4: How many committees can be formed with 3 professors from the 9-person math department and 4 professors from the 11-person CS department? $C(9,3) \cdot C(11,4) = 84 \cdot 330 = 27720$.

- **Permutations with repetition:** The number of r-permutations of a set of n elements with repetition allowed is n^r.

  Example 5: How many string of length 10 can be formed from the lower-case English alphabet? 26^{10}.

- **Combinations with repetition:** There are $C(n+r-1,r)$ r-combinations from a set with n types of unlimited number of elements when repetition is allowed.

  Why? We use stars $*$ to represent the elements selected and there are r stars. We use bars $|$ to create slots that represent the n types of elements and there are $n-1$ bars. Each sequence with r stars and $n-1$ bars (in any order) defines a way of selection. The number of such sequences is also the number of ways to place the r stars in $n-1+r$ positions, which is therefore $C(n+r-1,r)$ or $C(n+r-1,n-1)$.

  Example 6: How many different ways are there to buy six cookies from a store that sells four types of cookies? $C(4+6-1,6) = C(9,6) = C(9,3) = \frac{9\cdot 8\cdot 7}{3\cdot 2\cdot 1} = 84$.

  Example 7: How many nonnegative integer solutions does the equation $x_1 + x_2 + x_3 = 11$ have? This problem is equivalent to finding the number of sequences with 11 stars and 2 bars. So the total number is $C(11+2,2) = C(13,2) = \frac{13\cdot 12}{2\cdot 1} = 78$.

- **Permutation with indistinguishable objects:** The number of different permutations of n elements, where there are n_1 indistinguishable elements of type 1, n_2 indistinguishable elements of type 2, ..., and n_k indistinguishable elements of type k such that $n_1 + n_2 + \cdots + n_k = n$, is $\frac{n!}{n_1!n_2!\cdots n_k!}$.

  Why? The n_1 elements of type 1 can be placed among the n positions in $C(n,n_1)$ ways, leaving $n-n_1$ positions free. Then the n_2 elements of type 2 can be placed in $C(n-n_1,n_2)$ ways, leaving $n-n_1-n_2$ positions free. Continuing placing the elements of type 3, ..., type k, and by the product rule, we have $C(n,n_1)C(n-n_1,n_2)\cdots C(n-n_1-\cdots n_{k-1},n_k) = \frac{n!}{n_1!n_2!\cdots n_k!}$.

  Example 8: How many different strings can be made by reordering the letters of the word SUCCESS? $\frac{7!}{3!2!1!1!} = 420$.

- **Distributing objects into boxes:** Some counting problems can be modeled as enumerating the ways objects can be placed into boxes, where objects and boxes may be distinguishable or indistinguishable.

  - **Distinguishable objects and distinguishable boxes:** The number of ways to distribute n distinguishable objects into k distinguishable boxes so that n_i objects are placed into box i, where $i = 1,2,\ldots,k$, equals $\frac{n!}{n_1!n_2!\cdots n_k!}$.

Why? First choose n_1 objects from n to be placed in box 1, then choose n_2 objects from $n-n_1$ to be placed in box 2, and so on. By the product rule, the numbers of ways is $C(n,n_1)C(n-n_1,n_2)\cdots C(n-n_1-\cdots-n_{k-1},n_k) = \frac{n!}{n_1!n_2!\cdots n_k!}$.

Example 10: How many ways are there to distribute hands of 5 cards to each of four players from a standard deck of 52 cards? $\frac{52!}{5!5!5!5!32!}$.

- **Indistinguishable objects and distinguishable boxes:** The number of ways to distribute n indistinguishable objects into k distinguishable boxes is the same as the number of ways of choosing n objects from a set of k types of objects with repetition allowed, which is equal to $C(k+n-1,n)$.

  Example 11: How many ways are there to place 10 indistinguishable balls into 8 distinguishable bins? $C(8+10-1,10) = C(17,10) = \frac{17!}{10!7!} = 19448$.

- **Distinguishable objects and indistinguishable boxes:** This is a hard problem. We consider an example first.

  Example 12: How many ways are there to put 4 employees into three indistinguishable office? We name the employees by A,B,C,D. The question becomes: how many ways to partition set $\{A,B,C,D\}$ into three subsets?

  All four in one office: $0-0-4 \Rightarrow 1$

  Three in one office: $0-1-3 \Rightarrow 4$

  Two in one office and the other two in one office: $0-2-2 \Rightarrow 3$

  Two in one office and the other two each in one office: $1-1-2 \Rightarrow 6$

  So the finial answer is $1+4+3+6 = 14$.

  There are no simple closed formula for the number of ways to distribute n distinguishable objects into k indistinguishable boxes. However, there is a complicated formula. Let $S(n,j)$ be the number of ways to distribute n distinguishable objects into j indistinguishable boxes so that no box is empty. Therefore, the number of ways to distribute n distinguishable objects into k indistinguishable boxes is $\sum_{j=1}^{k} S(n,j)$. In the above example, $S(4,1) = 1, S(4,2) = 4+3 = 7, S(4,3) = 6$. It can be shown that

  $$S(n,j) = \frac{1}{j!}\sum_{i=0}^{j-1}(-1)^i \binom{j}{i}(j-i)^n.$$

  So

  $$\sum_{j=1}^{k} S(n,j) = \sum_{j=1}^{k}\frac{1}{j!}\sum_{i=0}^{j-1}(-1)^i \binom{j}{i}(j-i)^n.$$

- **Indistinguishable objects and indistinguishable boxes:**

  Example 13: How many ways are there to pack six copies of the same book into four identical boxes, where a box can hold as many as six books?

  We enumerate ways of packing: $(6),(5,1),(4,2),(3,3),(4,1,1),(3,2,1),(2,2,2),(3,1,1,1),(2,2,1,1)$. So the answer is 9.

  Distributing n indistinguishable objects into k indistinguishable boxes is the same as writing n as a sum of at most k positive integers in non-increasing order. There is no simple closed formula for this number.

5.4 Binomial coefficients

Reading: Rosen 5.4

Recall that $C(n,r)$ can also be denoted by $\binom{n}{r}$. It is sometimes called a **binomial coefficient** since it occurs as coefficient in the expansion of powers of binomial expressions such as $(x+y)^n$.

- **The Binomial Theorem:** Let x and y be variables and let n be a nonnegative integer. Then

$$(x+y)^n = \sum_{i=0}^{n}\binom{n}{i}x^{n-i}y^i = \binom{n}{0}x^n + \binom{n}{1}x^{n-1}y + \binom{n}{2}x^{n-2}y^2 + \cdots + \binom{n}{n-1}xy^{n-1} + \binom{n}{n}y^n.$$

Proof: Write $(x+y)^n$ as $(x+y)(x+y)\cdots(x+y)$. Each term in the product when it is expanded is of the form $x^{n-i}y^i$ for $i=0,1,2,\ldots,n$. To count the number of terms of the form $x^{n-i}y^i$, note that to obtain such a term it is necessary to choose $n-i$ xs from the n factors of $(x+y)$ in the product so that there are i ys in the term. Therefore, the coefficient of $x^{n-i}y^i$ is $\binom{n}{n-i}$, which equals $\binom{n}{i}$.

Example 1: What is the expansion of $(x+y)^4$?

$(x+y)^4 = \binom{4}{0}x^4 + \binom{4}{1}x^3y + \binom{4}{2}x^2y^2 + \binom{4}{1}xy^3 + \binom{4}{4}y^4 = x^4 + 4x^3y + 6x^2y^2 + 4xy^3 + y^4$.

Example 2: What is the coefficient of $x^{12}y^{13}$ in the expansion of $(2x-3y)^{25}$?

$\binom{25}{13}(2x)^{25-13}(-3y)^{13} = \binom{25}{13}2^{12}(-3)^{13}x^{12}y^{13}$. So the coefficient is $\binom{25}{13}2^{12}(-3)^{13}$.

- **Corollaries:** (1) $\sum_{k=0}^{n}\binom{n}{k} = 2^n$. (2) $\sum_{k=0}^{n}(-1)^k\binom{n}{k} = 0$. (3) $\sum_{k=0}^{n}2^k\binom{n}{k} = 3^n$.
- Identities of binomial coefficients:

  Pascal's identity: $\binom{n+1}{k} = \binom{n}{k-1} + \binom{n}{k}$ for positive integers $n \geq k$.

  Vandermonde's identity: $\binom{m+n}{r} = \sum_{k=0}^{r}\binom{m}{r-k}\binom{n}{k}$ for nonnegative integers m, n, r with r not exceeding either m or n.