Mixed-Weight Open Locating-Dominating Sets

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***Abstract*— Wireless sensor networks (WSNs) and other monitoring systems often use sensors of multiple strengths within the same network. This information is lost when the network is modeled by a typical non-weighted graph. We introduce the mixed-weight open locating-dominating set (mixed-weight OLD-set), an extension of the open locating-dominating set, that allows vertices in the graph to have different weights. We show that finding the minimum mixed-weight OLD-set is NP-complete, establish probabilistic bounds for the size of mixed-weight OLD-sets in random graphs, and provide simulation results supporting those bounds.**

I. Introduction

Location detection problems cover a wide range of applications including wireless sensor networks (WSNs), contaminant detection in public resources, and fault detection in microprocessors. These systems are modeled as a graph $G = (V, E)$, where V, the set of vertices, represents a set of specific locations in the system and E, the set of edges, is determined by the proximity of those locations (in wireless networks) or by physical connection. In general, location detection problems are used to find the smallest set of sensors needed to uniquely locate anomalies in a network.

Location detection problems offer many benefits in the design of WSNs including ease of monitoring and efficiency [11]. WSNs have been used to monitor a variety of environmental fields including volcanoes [15], glaciers [13], and animal behavior [12]. Location detection problems have been studied for contaminant detection and source location in public utilities and building ventilation systems [2], [3]. Open locating-dominating sets (OLD-sets) are of particular interest for situations in which a perpetrator releases a contaminant into a system and destroys a sensor at the release site. In WSNs, OLD-sets can also represent sensors that run extra processes to track the health of the entire system by monitoring and testing incoming messages.

Many studies have focused on identifying codes (ID-codes) and locating-dominating sets [4], [9]–[11], which are subsets of vertices that uniquely locate other vertices in a graph. If sensors are placed in the locations represented by those vertices, those sensors will be able to detect and locate problems in the system. OLD-sets, defined in [14], are an extension of ID-codes that assume if a problem occurs at the location of a sensor, the sensor will malfunction. Similarly, OLD-sets can represent systems which monitor their health by tracking messages received at particular locations.

In this paper we introduce the mixed-weight open locating-dominating set (mixed-weight OLD-set), an extension of the OLD-set that allows multiple integer weights to be given to vertices. The mixed-weight OLD-set problem models a system in which sensors of different strengths, and potentially different costs, are strategically placed throughout the system. An increase in the weight expands the reach of the vertex by an equivalent number of edges in the graph. For instance, a weight 1 vertex can reach its neighbors, a weight 2 vertex can reach its neighbors and its neighbors' neighbors, and so on. WSNs often use multiple types of sensors, such as in systems that monitor natural habitats [12]. Thus mixed-weight OLD-sets can aid in the development and cost management of these networks.

Mixed-weight OLD-sets are related to the weighted or d-identifying code, where all vertices receive the same weight d, which have been studied for paths and cycles [4], hypercubes [9], and other graphs. The mixed-weight OLD-set is also similar to an OLD-set on a directed graph: increased weight can be represented by arcs to other vertices. Identifying codes have also been studied for directed graphs in [10]. We note that, to our knowledge, there is no literature for weighted OLD-sets or OLD-sets in directed graphs. The weighted OLD-set is a special case of the mixed-weight OLD-set, thus this is the first time weighted or mixed-weight OLD-sets have been studied.

Random graphs have been studied in areas where the structure of the graph is unknown such as neural networks, social connections, the World Wide Web [5], and WSNs [7]. For WSNs, sensors are dispersed from aircraft and then connect to each other after distribution [1], thus the structure of the network is unknown until after deployment. Identifying codes were studied in random graphs in [8]. We expand those results to include (mixed-weight) open locating-dominating sets and provide simulation results to show the accuracy of the theoretical bounds. To our knowledge, this is the first time OLD-sets have been studied in random graphs.

This paper is organized as follows. In Section II we provide basic definitions and introduce the concept of mixed-weight open locating-dominating sets. In Section III we define several optimization problems related to mixed-weight OLD-sets and discuss their NP-completeness. We determine a probabilistic upper bound for mixed-weight OLD-sets in random graphs and provide simulation results that support the findings in Section IV. We conclude in Section V.

II. Mixed-Weight Open Locating-Dominating Sets

We begin with basic definitions of a neighbor and open locating-dominating sets and, then, introduce mixed-weight open locating-dominating sets. The *neighbor* of a vertex x is any vertex adjacent to x in a graph $G = (V, E)$. The *neighborhood* of a vertex x, $N(x)$ is the set of all neighbors of x, and does not include x (the open property).

An *open locating-dominating set* or *OLD-set*, is a set of vertices $S \subseteq V$ such that $N(x) \cap S$ is nonempty and unique for every $x \in V$, i.e., $\forall x \in V$, $N(x) \cap S \neq \varnothing$ (the dominating property) and $\forall x, y \in V$ with $x \neq y$, $N(x) \cap S \neq N(y) \cap S$ (the locating property).

The distance between two vertices, $d(x, y)$, is the length of the shortest path between vertex x and vertex y, where each edge is considered to be length 1.

Definition 1. The *weight function*, w, such that $w(x) \geq 1$ $\forall x \in V$, provides an integer value, or *weight*, for each vertex in the graph.

For a particular vertex x, $w(x)$ is the weight of the vertex. For a set of vertices $A \subseteq V$, the weight of the set $w(A)$ is the sum of all the weights of vertices in that set, $\sum_{x \in A} w(x)$.

Definition 2. The *open outgoing-ball*, $B^-(x)$ is the set of all vertices within a distance of $w(x)$ from x but not including x, i.e., $B^-(x) = \{y \in V | 0 < d(x, y) \leq w(x)\}$.

Definition 3. The *open incoming-ball*, $B^+(x)$ is the set of all vertices that contain x in their open outgoing-ball, i.e., $B^+(x) = \{y \in V | 0 < d(x, y) \leq w(y)\}$.

Definition 4. A *mixed-weight open locating-dominating set*, *mixed-weight OLD-set*, or *MW-OLD-set* of a graph G is a set of vertices $S \subseteq V$ such that $B^+(x) \cap S$ is nonempty and unique for every $x \in V$.

The case in which $w(x) = 1$ for all $x \in V$ defines the non-weighted OLD-set. For the mixed-weight OLD-set, we say a vertex y is a *neighbor* of vertex x if $y \in B^+(x)$. In this case neighbors are not symmetric. We note that the weight of a vertex x not in mixed-weight OLD-set S does not affect S or the size of S, both theoretically and in the application of sensor monitoring.

Definition 5. The *total weight* of the (mixed-weight) OLD-set S, $w(S)$, is the sum of all the weights of all vertices in S.

In Figure 1, a vertex with weight greater than 1 becomes the neighbor of other vertices. In a WSN, a weighted vertex under this definition would represent a sensor that has a stronger antenna and can therefore receive data from transmitters that are further away.

The following lemmas provide the necessary conditions for the existence of mixed-weight OLD-sets.

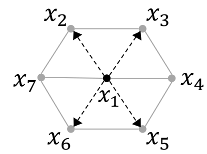
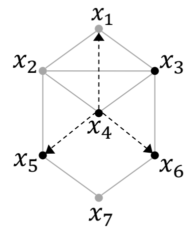


Fig. 1. The effect of the weighted vertex, $w(x_1) = 2$, is indicated by dashed arrows.



(a)

x	$B^+(x)$, weighted
x_1	$\{x_2, x_3, x_4\}$
x_2	$\{x_1, x_3, x_4, x_5\}$
x_3	$\{x_1, x_2, x_4, x_6\}$
x_4	$\{x_2, x_3\}$
x_5	$\{x_2, x_4, x_7\}$
x_6	$\{x_3, x_4, x_7\}$
x_7	$\{x_5, x_6\}$

(b)

Fig. 2. Mixed-Weight OLD-set Example: If there are no weighted vertices, x_1 and x_4 share open incoming-ball $\{x_2, x_3\}$, and the graph does not have an OLD-set. If $w(x_4) = 2$, as indicated by dashed arrows, then $\{x_3, x_4, x_5, x_6\}$ is one minimum mixed-weight OLD-set, as shown with black vertices.

Lemma 6. *For graph $G = (V, E)$ and weight function w, a mixed-weight OLD-set $S \subseteq V$ exists if and only if $\forall x, y \in V$, $x \neq y$, $B^+(x) \neq B^+(y)$, i.e., if the open incoming-ball is unique for all vertices in the graph.*

Lemma 7. *For graph $G = (V, E)$ and weight function w, if a mixed-weight OLD-set exists, then $S = V$ is a mixed-weight OLD-set.*

Lemma 8. *For every connected graph with $|V| > 1$, there exists a weight function such that the graph contains a mixed-weight OLD-set. Specifically, if $w(x)$ is the greatest distance between x and any other vertex in the graph, known as the eccentricity of x, then the graph always has a mixed-weight OLD-set.*

Figure 2 shows an example of a mixed-weight OLD-set in a graph that does not contain a non-weighted OLD-set. Vertices x_1 and x_4 share open incoming-ball $\{x_2, x_3\}$ when $w(x) = 1$ $\forall x \in V$. If vertex x_4 is given weight 2, then every open incoming-ball becomes unique, allowing for a mixed-weight OLD-set. One minimum sized mixed-weight OLD-set is $\{x_3, x_4, x_5, x_6\}$.

We note that the mixed-weight identifying code is defined similarly to the mixed-weight OLD-set. An *identifying code* is a set of vertices, $S \subseteq V$, such that every vertex in the graph has a unique and non-empty set of neighbors in S, where a vertex x is considered a neighbor of itself. The *mixed-weight identifying code* is a set of vertices $S \subseteq V$ such that the *closed incoming-ball*, $B^+[x]$, is non-empty and unique for every $x \in V$. The closed incoming-ball is simply $B^+[x] = B^+(x) \cup \{x\}$.

III. Problems and NP-Completeness

We introduce three optimization problems based on the mixed-weight OLD-set. These problems are finding the minimum size of a mixed-weight OLD-set for a graph and weight function, finding the weight function that produces the minimum mixed-weight OLD-set for a graph, and finding the minimum total weight of the mixed-weight OLD-set. We show that the decision problem for determining the minimum size of the mixed-weight OLD-set is NP-Complete for weights 1 through d, for any positive integer d.

A. Minimum Size of the Mixed-weight OLD-set

Consider a graph $G = (V, E)$ and a weight function w. We want to determine the smallest set S that is a mixed-weight OLD-set. Alternatively, we can ask the question, for graph $G = (V, E)$, weight function w, and positive integer $k \leq |V|$, is there a mixed-weight OLD-set S where $|S| \leq k$?

Example 9. Consider a cycle of size 10 with vertices labeled $\{0, 1, 2 \dots 9\}$ and weight function $w(x) = 2$ for $x \in \{0, 5\}$ and $w(x) = 1$ otherwise. The minimum size of the mixed-weight OLD-set is 6 and one satisfying mixed-weight OLD-set is $S = \{0, 1, 2, 5, 6, 7\}$.

The decision problem for finding the minimum mixed-weight OLD-set in a graph is as follows.
NAME: Mixed-weight open locating-domination set (MW-OLD).
INSTANCE: A connected graph $G = (V, E)$, weight function w, and integer $k \leq |V|$ is a positive integer.
QUESTION: Is there a mixed-weight OLD-set $S \subseteq V$ of size at most k?.

We prove that this decision problem is NP-Complete, requiring the following lemmas.

Lemma 10. *Let $P = \{p_1, p_2, p_3\}$, $Q = \{q_1, q_2, q_3\}$, and Δ be a subgraph of a graph $G = (V, E)$ with vertex set $V_\Delta = P \cup Q \cup \{r\}$ and edge set $E_\Delta = \{(q_i, p_i), (q_i, q_j), (q_i, r) | 1 \leq i, j \leq 3, i \neq j\}$, as show in Figure 3a, and suppose that r is the only vertex in V_Δ that is adjacent to another vertex in G. If all vertices in $V - (P \cup Q)$ are not in the open incoming-ball of the vertices in P, then Q must be in any mixed-weight OLD-set of G.*

Proof: The vertices in P are endpoints, therefore Q must be in the mixed-weight OLD-set S of G so that S is dominating. ■

Lemma 11. *Let P, Q, and Δ be as defined in Lemma 10. If r is the only vertex in $V - (P \cup Q)$ that is in the open incoming-ball of the vertices in P, then Q must be in the minimum mixed-weight OLD-set of G.*

Proof: First, we note that if Q is in the mixed-weight OLD-set S, then the vertices in Δ are open located and dominated. WLOG, suppose that q_1 is not in S. Then p_3 must be in S, so p_2 and q_3 are open located and dominated, p_2 must be in S so p_3 and q_2 are open located and dominated, and r must be in S so p_1 is open located and dominated. Thus the minimum mixed-weight OLD-set of G must contain Q. ■

Lemma 12. *Let P, Q, and Δ be as defined in Lemma 10. If $x \neq r$ is the only vertex in $V - (P \cup Q)$ that is in the open incoming-ball of the vertices in P, then Q must be in the minimum mixed-weight OLD-set of G.*

Proof: First, we note that if Q is in the mixed-weight OLD-set S, then the vertices in Δ are open located and dominated. WLOG, suppose that q_1 is not in S. Then r must be in S, so p_2, p_3, q_2, and q_3 are open-located and dominated, and x must be in S so p_1 is open located and dominated. Thus the minimum mixed-weight OLD-set of G must contain Q. ■

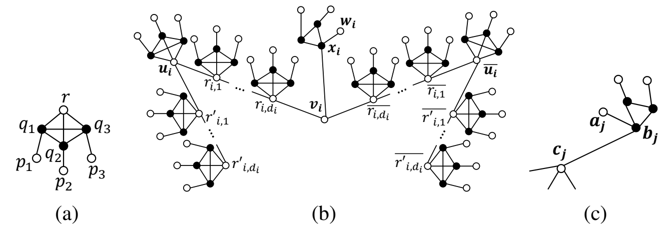


Fig. 3. The Δ subgraph, literal component H_j, and clause component G_i for MW-OLD.

We now prove the decision problem for finding the minimum mixed-weight OLD-set in a graph with weight function w and $d = \max\{w(x) | x \in V\}$, is NP-Complete.

Theorem 13. *MW-OLD is NP-Complete.*

Proof: First we show that MW-OLD is in NP. Given a subset $S \subseteq V$, constructing $B^+(x) \cap S$ for each $x \in V$ can be done in polynomial time on $|V|$ since finding the shortest path and the intersection can be done in polynomial time. Therefore, determining that $B^+(x) \cap S$ is unique and nonempty for $x \in V$ can also be done in polynomial time on $|V|$.

Next, we polynomially reduce 3-SAT to MW-OLD, inspired by the proof of OLD in [14]. Given an instance of 3-SAT with literals $U = \{u_1, u_2 \dots u_N\}$ and clauses $C = \{c_1, c_2 \dots c_M\}$ we construct a literal component G_i, $1 \leq i \leq N$, for each literal as seen in Figure 3b, and a clause component H_j, $1 \leq j \leq M$, for each clause as shown in Figure 3c. Let Δ be a subgraph as shown in Figure 3a.

Let $d_i = i \mod d$, and let $r_{i,0}, r'_{i,0} = u_i$ and $\overline{r_{i,0}}, \overline{r'_{i,0}} = \overline{u_i}$. In the literal component G_i, there is a set of d_i Δ subgraphs between v_i and u_i, with v_i adjacent to r_{i,d_i}, and $r_{i,k}$ adjacent to $r_{i,k+1}$ for $0 \leq k < d_i$. There is another set of d_i Δ subgraphs between v_i and $\overline{u_i}$ with adjacency defined similarly for $\overline{r_{i,k}}$, $0 \leq k < d_i$. There are two more sets of d_i Δ subgraphs, where $r'_{i,k}$ is adjacent to $r'_{i,k+1}$, and $\overline{r'_{i,k}}$ is adjacent to $\overline{r'_{i,k+1}}$, $0 \leq k < d_i$. Vertex v_i is also adjacent to the subgraph containing x_i and w_i, and vertices u_i and $\overline{u_i}$ are each a part of their own Δ subgraph.

In the clause component H_j, vertex c_j is adjacent to subgraph containing b_j and a_j. For clause C_j, $1 \leq j \leq M$, with literals $u_{j,1}, u_{j,2}, u_{j,3}$, connect vertex c_j to $r_{j,\ell}$ where $r_{j,\ell}$ is r'_{i,d_i} if $u_{j,\ell} = u_i$ or $r_{j,\ell}$ is $\overline{r'_{i,d_i}}$ if $u_{j,\ell} = \overline{u_i}$, for $1 \leq \ell \leq 3$.

We also construct weight function $w(x) = d_i + 1$, i.e. $1 \leq w(x) \leq d$, if x is a vertex representing the literal u_i or its complement $\overline{u_i}$, and $w(x) = 1$ otherwise. Given this graph and weight function, we have constructed a graph $G = (V, E)$ with $|V| = 7M + 21N + 28 \sum_{i=1}^{N} d_i$ and $|E| = 10M + 27N + 40 \sum_{i=1}^{N} d_i$, and this can be done in polynomial time. We also set $k = 3M + 10N + 3 \sum_{i=1}^{N} d_i$.

If $S \subseteq V$ is a mixed-weight OLD-set on G, then S must contain every vertex adjacent to a vertex of degree 1, to satisfy the dominating property of the mixed-weight OLD-set. By construction, if a literal or its complement is in S, then, by Lemmas 10, 11, and 12, for vertices in G that form a Δ subgraph it is best to include the vertices equivalent to q_1, q_2, and q_3, as shown in Figure 3a, in S. Thus a minimum construction of S must contain all the black vertices shown in Figure 3 for a total of $3M + 9N + 3\sum_{i=1}^{N} d_i$. To satisfy the locating property of the mixed-weight OLD-set, $B^+(w_i) \neq B^+(v_i)$, so at least one of u_i, $\overline{u_i}$, r_{i,d_i}, and $\overline{r_{i,d_i}}$ must be in S. Similarly, $B^+(a_j) \neq B^+(c_j)$, so at least one of $u_{j,1}, u_{j,2}$, $u_{j,3}$, $r_{j,1}$, $r_{j,2}$, $r_{j,3}$ must be in S. Thus given $k = 3M + 10N + 3\sum_{i=1}^{N} d_i$, it is clear that C is satisfiable if and only if $|S| = k$. ■

Corollary 14. *The decision problem for finding the minimum MW-OLD-set in a graph with weight function with d possible weights, including weight 1, i.e. $w(x) \in \{w_1, w_2, \dots w_d\}$, $\forall x \in V$, where w_h is a positive integer for $1 \leq h \leq d$ and $\exists h$ where $w_h = 1$, is NP-Complete.*

Proof: Define $d_i = w_{i'} - 1$, where $i' = i \mod d + 1$, so $d_i \in \{w_1 - 1, w_2 - 1, \dots, w_d - 1\}$. Define the literal components G_i, the clause components H_j, and the weight function $w(x)$ as above, so $w(x) \in \{w_1, w_2, \dots w_d\}$ if x is a vertex representing the literal u_i or its complement $\overline{u_i}$, and $w(x) = 1$ otherwise. The rest of the proof follow similarly to the proof in Theorem 13. ■

Finding the minimum size of the mixed-weight OLD-set is applicable to both planned and randomly structured WSNs. For WSNs designed before deployment and placed in specific locations, the minimum-mixed weight OLD-set provides the simplest and smallest structure for the entire system. Randomly dispersed WSNs are typically distributed over large areas and sometimes over unusual terrain, making day-to-day, manual upkeep of the network impractical. The minimum mixed-weight OLD-set in these large systems provides the smallest number of sensors that can track the health of the overall network of sensors, reducing the computational and battery power needed for tasks not associated with network's main purpose.

B. Minimizing the Size of a Mixed-weight OLD-set

To minimize the size of a mixed-weight OLD-set, we are given a graph $G = (V, E)$, and we want to determine a weight function w that gives a minimum sized mixed-weight OLD-set. Alternatively, we can ask the question, for graph $G = (V, E)$ and positive integers $j, k \leq |V|$, is there a weight function w, with $\max(w) = j$, that allows for a mixed-weight OLD-set S such that $|S| \leq k$?

Example 15. Consider a cycle of size 15 with vertices labeled $\{0, 1, 2 \dots 14\}$. One minimizing weight function for $\max(w) = 2$, is $w(x) = 2$ for $x \in \{3, 5, 7, 9, 11\}$ and $w(x) = 1$ otherwise. The minimum OLD-set size is 7, for example, $S = \{1, 3, 5, 7, 9, 11, 13\}$. The minimum OLD-set when $w(x) = 2$ for all x is also size 7, and the minimum OLD-set with $w(x) = 1$ for all x is size 10. Thus in this particular graph, a mixed-weight OLD-set provides a smaller sized OLD-set than using all weight 1 vertices, and provides the same minimum sized OLD-set when using all weight 2 vertices.

A particular application of this problem is in the design of WSNs. If we want to make the network as small as possible in order to reduce administrative cost, then a multitude of potential weight functions should be considered.

C. Minimizing the Total Weight of a Mixed-weight OLD-set

To minimize the total weight of a mixed-weight OLD-set, we are given a graph $G = (V, E)$ and weight function w, and we want to determine the minimum total weight of all the vertices in the mixed-weight OLD-set. Alternatively, we can ask the question, for graph $G = (V, E)$ and $k \leq |V|$, is there a weight function w that allows for a mixed-weight OLD-set S such that $w(S) \leq k$?

In many cases finding the minimum sized mixed-weight OLD-set also provides the minimum total weight, however, for graphs with multiple minimum-sized mixed-weight OLD-sets, the minimum-sized mixed-weight OLD-sets may provide different total weights.

Example 16. Consider a cycle of size 8 with vertices labeled $\{0, 1, 2 \dots 7\}$, and weight function $w(x) = 2$ for $x \in \{0, 3, 4, 7\}$, and $w(x) = 1$ otherwise. One minimum-sized mixed-weight OLD-set is $\{0, 3, 4, 7\}$ which results in a total weight of 8. However, the set $\{3, 4, 6, 7\}$ is also a mixed-weight OLD-set with a total weight of 7.

This problem is also applicable to WSNs. The total cost of the network could be reduced by replacing several cheaper, weaker sensors with a single stronger sensor. This could be extended by considering that the actual cost of a sensor is not necessarily proportional to its strength. In that case a cost function based on the weight could be introduced to the problem, with the goal of reducing the total cost of the system.

The following proposition gives a lower bound on the total weight of an mixed-weight OLD-set in a path.

Proposition 17. *A path with n vertices does not have a mixed-weight OLD-set with total weight $w(S) < \lceil (2/3)n \rceil$.*

Proof: From [14] we know that in a path with unweighted vertices, for every six consecutive vertices in the path, at least four of those vertices must be in the mixed-weight OLD-set S resulting in $|S| = w(s) \geq \lceil (2/3)n \rceil$. If there exists a weight function $w(x) \leq 2$ such that $w(S) < \lceil (2/3)n \rceil$, then at least one set of six consecutive vertices in the path $X = \{x_i, x_{i+1}, x_{i+2}, x_{i+3}, x_{i+4}, x_{i+5}\}$ must have $w(X \cap S) = 3$. It cannot be the case that $X \cap S$ has 3 weight 1 vertices. If $X \cap S$ contains one weight 2 vertex and one weight 1 vertex, then there are only 3 unique and nonempty subsets of $X \cap S$. Thus there are not enough unique, non-empty subsets to locate and dominate all six vertices in X. This result follows similarly for $w(x) \leq d$ for $d > 2$. ■

IV. Random Graphs

Many WSNs are distributed over a large area at random, with their structure unknown until deployment. Therefore the study of random graphs provides unique insight into these systems. Random graphs $G_{n,p}$ are graphs with n vertices where every edge exists with probability p and does not exist with probability $1 - p$ [6]. When generating such a graph, the probability of obtaining a particular graph with m edges is $p^m(1-p)^{\binom{n}{2}-m}$, which is the probability that m particular edges exist and the remaining edges do not exist. We study the mixed-weight OLD-set problem in random graphs with particular focus on bounds on the size of the mixed-weight OLD-set. We provide both theoretical and experimental results.

A. Mixed-Weight OLD-set Bounds in Random Graphs

We show the relationship between the probabilty of an edge existing in a random graph and the existence of a mixed-weight OLD-set in the following lemma. We also determine an upper bound on the size of a mixed-weight OLD-set in a random graph for any set of positive, integer weights up to weight d in the subsequent theorem. For random graphs, we say *almost every* graph in $G_{n,p}$ has property X if, as n goes to infinity, the probability that $G_{n,p}$ has property X goes to 1.

Lemma 18. *Almost every graph in $G_{n,p}$ has a mixed-weight OLD-set with $|S| \leq \frac{(2+\varepsilon)\log n}{\log 1/q}$, where q is the probability that a vertex is either in or not in both open incoming-balls of two other vertices. Values p, $1-p$, and ε are $\omega(1/\log n)$, i.e., the values are asymptotically greater than $1/\log n$.*

Proof: This result follows similarly to the result for identifying codes in [8]. ∎

We note that q provides the probability that a particular vertex does not help a mixed-weight OLD-set dominate two other vertices (is not in both incoming-balls) or does not help locate two other vertices (is in both incoming balls).

Theorem 19. *Almost every graph in $G_{n,p}$ has a mixed-weight OLD-set S with $|S| \leq \frac{(2+\varepsilon)\log n}{\log 1/q}$ for*

$$q = \left(\sum_{i=1}^{d} \rho_i \sum_{j=1}^{i} p^j(1-p^{j-1})\right)^2 + \left(\sum_{i=1}^{d} \rho_i \prod_{j=1}^{i}(1-p^j)\right)^2,$$

where $\rho_i \geq 0$ is the probability that a vertex is weight i, and $\sum_{i=1}^{d} \rho_i = 1$. Values p, $1-p$, and ε are $\omega(1/\log n)$, i.e., the values are asymptotically greater than $1/\log n$.

Proof: Let q be the probability that a vertex z is either in or not in both open incoming-balls of two other vertices $x \neq y$. If z is in the open incoming-ball of a vertex x, then if z is weight $i \leq d$, there is a path of length at most i between x and z with probability $\sum_{j=1}^{i} p^j(1-p^{j-1})$. If z is not in the open incoming-ball of a vertex x, then if z is weight $i \leq d$, there is no path of length at most i between x and z with probability $\prod_{j=1}^{i}(1-p^j)$. So the probability that z is in the open incoming-ball of x is $\sum_{i=1}^{d} \rho_i \sum_{j=1}^{i} p^j(1-p^{j-1})$, and the probability that z is not in the open incoming-ball of x is $\sum_{i=1}^{d} \rho_i \prod_{j=1}^{i} p^j(1-p^{j-1})$. ∎

B. Simulation Results

We generated random graphs and weight functions, with weights 1 and 2, and determined the actual size of their mixed-weight OLD-sets using an exhaustive search. These results were compared with bounds given in Section IV-A. Our results show that as n increases, the probability that a graph contains a mixed-weight OLD-set with size less than or equal to the smallest possible bound also increases.

Using a Lehmer random number generator, graphs $G_{n,p}$ with corresponding weight functions, $w(x) = 2$ with probability ρ and $w(x) = 1$ otherwise, were generated for each combination of $n \in \{10, 15, 20, 25\}$, $p \in [0.45, 0.55]$ for $n = 10$, $p \in [0.4, 0.6]$ for $n = 15$, $p \in [0.35, 0.65]$ for $n = 20$ and $n = 25$, $\rho \in [0.35, 0.65]$. Each p and ρ were incremented by 0.05, and both p and $1-p$ were $\omega(1/\log n)$, i.e., the values are greater than $1/\log n$. $10,000$ graphs and corresponding weight functions were generated for each n, p, and ρ combination.

Consider the upper bound on the mixed-weight OLD-set size from Theorem 19, $\frac{(2+\varepsilon)\log n}{\log 1/q}$, when ε was as small as possible, i.e. $\varepsilon = 1/\log n$, which we'll call the *minimum bound*. In Figure 4, the column titled 'MW-OLD-set' shows that as n increased the probability that a graph contained a mixed-weight OLD-set sharply increased, as expected from Lemma 18. Similarly the probability that a graph contains a mixed-weight OLD-set with size less than or equal to the minimum bound increased, as seen in the column titled '≤ Bound'. These trends show that the bound given in Theorem 19 is tight as n increases, even when ε is as small as possible.

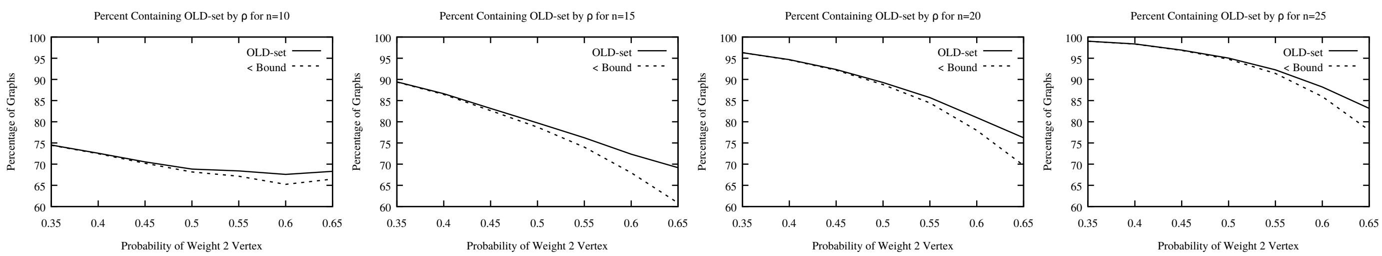
For each n, Figure 5 shows the percentage of graphs that had a mixed-weight OLD-set and the percentage that had a mixed-weight OLD-set with size less than or equal to the minimum bound given a value of ρ. Except for $n = 10$, as ρ increases, the probability of a mixed-weight OLD-set or a mixed-weight OLD-set with size less than or equal to the bound decreases. One possible explanation for this pattern is, given that there are fewer possible subsets of neighbors when n is small, there may be an increased likelihood that weighted vertices cause other vertices to share open incoming-balls. This explanation is also supported by Figure 5 as it appears that the effect of ρ is decreasing as n increases. It is likely $n = 10$ is too small for a pattern to be consistent. For comparison, we also considered the percentage of graphs that had a mixed-weight OLD-set and the percentage that had a mixed-weight OLD-set with size less than the minimum bound given a value of p. The value of p had almost no impact on the percentages, unlike the impact of ρ.

Our simulation results were constrained by the size of the graph and the subsequent time to find the smallest mixed-weight OLD-set. To test larger graphs, we would need to rely on heuristics to generate solutions that would not be provably minimal. Though WSNs often contain thousands of nodes, there are many instances of smaller scale networks such as those simulated. Previously, 16 nodes monitored seismic activity on a volcano [15], 32 nodes monitored the habitat of Great Duck Island [12], and 8 nodes monitored glacial movement [13].

These simulation results show that for most randomly distributed WSNs, a small mixed-weight OLD-set can be found among the sensors, allowing this small number of sensors to monitor the health of the overall system. The results also show that the bound given in Section IV-A is tight for most graphs. This bound can provide WSN developers with an understanding of the number of sensors needed to track

Simulation Setup			Simulation Results				
n	Range of p	Range of ρ	Connected	MW-OLD-set	Size	Min. Bound	$\leq$ Bound
10	$[0.45, 0.55]$	$[0.35, 0.65]$	97.5%	70.1%	5.51	8.39	69.2%
15	$[0.40, 0.60]$	$[0.35, 0.65]$	99.7%	79.5%	6.40	9.85	77.2%
20	$[0.35, 0.65]$	$[0.35, 0.65]$	99.9%	88.8%	6.86	11.18	86.3%
25	$[0.35, 0.65]$	$[0.35, 0.65]$	99.9%	93.3%	7.03	11.89	92.1%

Fig. 4. Simulation trends for graphs graphs $G_{n,p}$ and weight function, $w(x) = 2$ with probability ρ and $w(x) = 1$ otherwise. Both p and ρ were incremented by 0.05 within their range. 'Connected' is the percentage of graphs that were connected (graphs do not need to be connected to contain a mixed-weight OLD-set). 'MW-OLD-set' is the percentage of graphs that contained a mixed-weight OLD-set. 'Size' is the average minimum size of the mixed-weight OLD-set if the graph contained one. 'Min. Bound' is the average upper bound of the mixed-weight OLD-set size $\frac{(2+\varepsilon)\log n}{\log 1/q}$ when $\varepsilon = 1/\log n$. '$\leq$ Bound' is the percentage of graphs that had a mixed-weight OLD-set with size less than or equal the minimum bound.



(a) (b) (c) (d)

Fig. 5. For each n, the percentage of graphs containing a mixed-weight OLD-set (solid line) and containing mixed-weight OLD-set with size less than or equal to the smallest possible bound (dashed line) are graphed by ρ, the probability that a vertex is weight 2. Values were averaged across p.

network health, given the overall size of the WSN, before the system is deployed.

V. Conclusion

In this paper we introduced mixed-weight open locating-dominating sets, an extension of open locating-dominating sets. The mixed-weight OLD-set addresses a unique problem in wireless sensor networks and other monitoring systems. Previously no model has been provided for systems that use sensors of varying strengths. Similar to other location detection problems, we showed that finding the minimum mixed-weight OLD-set is NP-Complete.

Given the unknown nature of many WSNs, studying location detection problems in random graphs is of particular interest. We showed that almost all random graphs have a mixed-weight OLD-set below a certain bound with both theoretical and simulation results. These results have the potential to help determine best practices in WSN design. To our knowledge this is the first time (mixed-weight) OLD-sets have been studied in random graphs.

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