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# TIGHT WORST-CASE PERFORMANCE BOUNDS FOR NEXT-$k$-FIT BIN PACKING*

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**Abstract.** The bin packing problem is to pack a list of reals in (0, 1] into unit-capacity bins using the minimum number of bins. Let $R[A]$ be the limiting worst value for the ratio $A(L)/L^*$ as $L^*$ goes to $\infty$, where $A(L)$ denotes the number of bins used in the approximation algorithm $A$, and $L^*$ denotes the minimum number of bins needed to pack $L$. Obviously, $R[A]$ reflects the worst-case behavior of $A$. For Next-$k$-Fit($NkF$ for short, $k \geq 2$), which is a linear time approximation algorithm for bin packing, it was known that $1.7 + \frac{3}{10(k-1)} \leq R[NkF] \leq 2$. In this paper, a tight bound $R[NkF] = 1.7 + \frac{3}{10(k-1)}$ is proved.

**Key words.** bin packing, approximation algorithm, worst-case performance

**AMS(MOS) subject classifications.** 68Q25, 68R05

**1. Introduction.** Given a finite list $L = (a_1, a_2, \ldots, a_m)$ of reals in (0, 1], and a sequence of unit-capacity bins, $B_1, B_2, \ldots$, the bin packing problem is to pack the numbers in the list into the bins such that no bin contains a total exceeding 1 and that the number of bins used is minimized.

Since the bin packing is NP-complete [9], no polynomial-time algorithm has ever been developed. A lot of effort has been made to find good approximation algorithms for the problem.

In order to evaluate and compare the quality of different approximation algorithms, we need to have a rigorous mathematical analysis of the worst-case behavior of these algorithms. Given an approximation algorithm $A$, and for any list $L$, let $A(L)$ be the number of bins used in the packing resulting when $A$ is applied to $L$, and $L^*$ be the minimum number of bins needed to pack $L$. The *worst-case performance bound* of the approximation algorithm $A$ is defined to be $R[A] = \limsup \max\{A(L)/L^*\}$ as $L^* \to \infty$.

Besides those well-studied approximation algorithms such as First-Fit ($FF$), Best-Fit ($BF$), First-Fit-Decreasing ($FFD$), Best-Fit-Decreasing ($BFD$), and Next-Fit ($NF$) [1], [5], [6], [7], [8], there is another important algorithm called Next-$k$-Fit ($NkF$), where $k$ is an integer greater than 1. In $NkF$, we process the numbers in $L$ in turn, starting from $a_1$, which is placed at the bottom of the first bin $B_1$. Suppose that $a_i$ is now to be packed. We look at the last $k$ nonempty bins. If $a_i$ does not fit into any of them, a new bin is created; otherwise, $a_i$ will go to the lowest indexed one of these $k$ nonempty bins into which it fits. Earlier, Johnson [7] proved that $1.7 + \frac{3}{10k} \leq R[NkF] \leq 2$. In the recent paper written by Csirik and Imreh [2], a new lower bound of $R[NkF]$ was given. They showed that $R[N2F] = 2$ and $1.7 + \frac{3}{10(k-1)} \leq R[NkF] \leq 2$ for $k \geq 3$. In this paper, we study the tight worst-case performance bound for the Next-$k$-Fit algorithm. Our result is the following theorem.

MAIN THEOREM.

$$R[NkF] = 1.7 + \frac{3}{10(k-1)}, \quad k \geq 2.$$

In §2, we study the upper bound proving technique for $NkF$ bin packing. In §3, we prove an important lemma. In §4, we show the proof of the main theorem.

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**2. The upper bound of $R[NkF]$.** It is known that $R[N2F] = 2$, and $R[NkF] \geq 1.7 + \frac{3}{10(k-1)}$ for $k \geq 3$ [2]. To prove the main theorem, all we need to do is prove the upper bound result, i.e., $R[NkF] \leq 1.7 + \frac{3}{10(k-1)}$ for $k \geq 3$. We need some careful analyses and preliminary results.

In the $NkF$ packing of any list $L$, there are $NkF(L)$ nonempty bins, $B_1, B_2, \ldots, B_{NkF(L)}$. For each bin $B_i$, its content can be divided into $k$ areas, $A_{i,1}, A_{i,2}, \ldots, A_{i,k}$, where $A_{i,1}$ contains all the numbers coming to $B_i$ when $B_i$ is the rightmost, or, in other words, the most recently created nonempty bin in the current packing, and $A_{i,2}$ contains all the numbers coming to $B_i$ when $B_i$ becomes the second rightmost nonempty bin, etc. Finally, $A_{i,k}$ contains all the numbers coming to $B_i$ when $B_i$ becomes the oldest among the $k$ active bins and is about to be thrown away. Figure 1 shows the division for $N3F$.

To prove the upper bound, we wish to show that $NkF(L) \leq (1.7 + \frac{3}{10(k-1)})L^* + c$ for all $L$, where $c$ is a constant. With the help of the following weighting function $W$ : $(0,\ 1] \rightarrow R^+$, also shown in Fig. 2, we will find the relation between $NkF(L)$ and $L^*$.

$$W(\alpha) = \begin{cases} \frac{6}{5}\alpha & \text{if } \alpha \in (0,\ \frac{1}{6}]; \\ \frac{9}{5}\alpha - \frac{1}{10} & \text{if } \alpha \in (\frac{1}{6},\ \frac{1}{3}]; \\ \frac{6}{5}\alpha + \frac{1}{10} & \text{if } \alpha \in (\frac{1}{3},\ \frac{1}{2}]; \\ \frac{6}{5}\alpha + \frac{2}{5} + \frac{3}{10(k-1)} & \text{if } \alpha \in (\frac{1}{2},\ 1]. \end{cases}$$

For any number $a_i$ in $L$, $W(a_i)$ is called the weight of $a_i$. $W(B_i)$, the weight of the bin $B_i$, is defined to be the sum of the weight of all numbers in $B_i$, i.e., $W(B_i) = \sum_{\forall a_j \in B_i} W(a_j)$. And $W(L)$, the weight of the list $L$, is defined to be the sum of the weight of all numbers in $L$, i.e., $W(L) = \sum_{\forall a_j \in L} W(a_j)$. When there is no possibility of confusion, we also use $B_i$ to denote the sum of the numbers in bin $B_i$, $A_{i,h}$ the sum of the numbers in area $A_{i,h}$, and $b_i$ the bottommost item in bin $B_i$.

**3. A lemma.**

LEMMA. *In the $NkF$ packing of $L$, for $j < NkF(L)$, if $B_j < \frac{5}{6}$, then there is $l > 0$ such that either* (1) $j + l \leq NkF(L)$, *and* $\frac{6}{5}B_j + W(B_{j+1}) + \cdots + W(B_{j+l}) \geq l + \frac{6}{5}B_{j+l}$, *or* (2) $j + l = NkF(L)$, *and* $\frac{6}{5}B_j + W(B_{j+1}) + \cdots + W(B_{NkF(L)}) + 2 \geq l + \frac{6}{5}B_{NkF(L)}$.

*Proof.* For notational simplicity, we assume $j = 1$. Because $B_1 < \frac{5}{6}$, items in $A_{2,1}$ and $A_{3,1}$ must be greater than $\frac{1}{6}$. Consider the following cases.

*Case* I. If $B_1 \leq \frac{1}{2}$, then $B_1$ must be followed by $k$ bins with their bottommost items greater than $\frac{1}{2}$, i.e., $b_2, \ldots, b_{k+1} > \frac{1}{2}$ (Fig. 3).

$$\begin{aligned} &\tfrac{6}{5}B_1 + W(B_2) + \cdots + W(B_k) + W(B_{k+1}) \\ &\geq \tfrac{6}{5}A_{1,1} + (\tfrac{6}{5}b_2 + \tfrac{2}{5} + \tfrac{3}{10(k-1)}) + \cdots + (\tfrac{6}{5}b_k + \tfrac{2}{5} + \tfrac{3}{10(k-1)}) + \tfrac{6}{5}B_{k+1} + \tfrac{2}{5} + \tfrac{3}{10(k-1)} \\ &\geq \tfrac{6}{5}(A_{1,1} + b_2) + \tfrac{6}{5}(b_3 + \cdots + b_k) + (\tfrac{2}{5} + \tfrac{3}{10(k-1)})k + \tfrac{6}{5}B_{k+1} \\ &\geq \tfrac{6}{5} \times 1 + \tfrac{6}{5} \times \tfrac{1}{2} \times (k-2) + (\tfrac{2}{5} + \tfrac{3}{10(k-1)})k + \tfrac{6}{5}B_{k+1} \\ &\geq k + \tfrac{6}{5}B_{k+1}. \end{aligned}$$

*Case* II. If $\frac{1}{2} < B_1 < \frac{5}{6}$, then we consider the cases in Fig. 4.

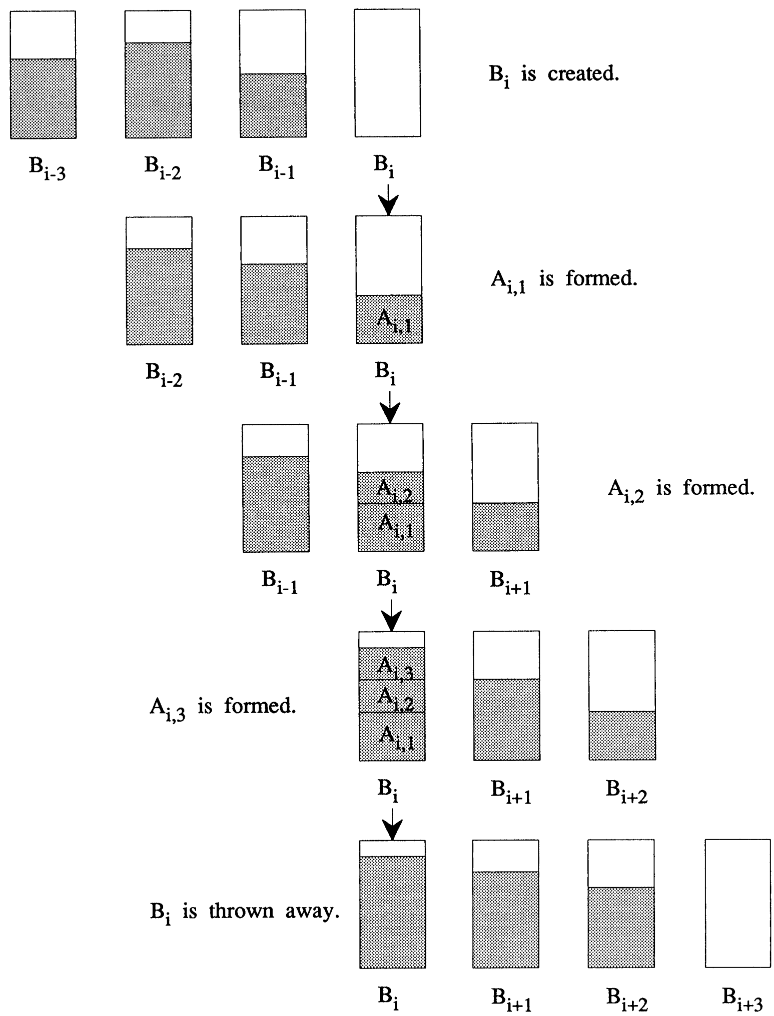


FIG. 1. *How the three areas of $B_i$ in $N3F$ packing are formed.*

*Case* 1. $B_2$ has one item greater than $\frac{1}{2}$. We have

$$\begin{aligned}
&\tfrac{6}{5}B_1 + W(B_2) \\
&\quad \geq \tfrac{6}{5}B_1 + \tfrac{6}{5}B_2 + \tfrac{2}{5} + \tfrac{3}{10(k-1)} \\
&\quad \geq \tfrac{6}{5} \times \tfrac{1}{2} + \tfrac{2}{5} + \tfrac{6}{5}B_2 \\
&\quad \geq 1 + \tfrac{6}{5}B_2.
\end{aligned}$$

Starting from now, we assume that all the items in $B_2$ are no greater than $\frac{1}{2}$.

*Case* 2. $A_{2,1} > \frac{1}{2}$. Since $A_{2,1}$ has at least two items, we assume at least one of its two bottommost items is in $(\frac{1}{6}, \frac{1}{3}]$. It is clear that $B_1$ is greater than $\frac{2}{3}$.

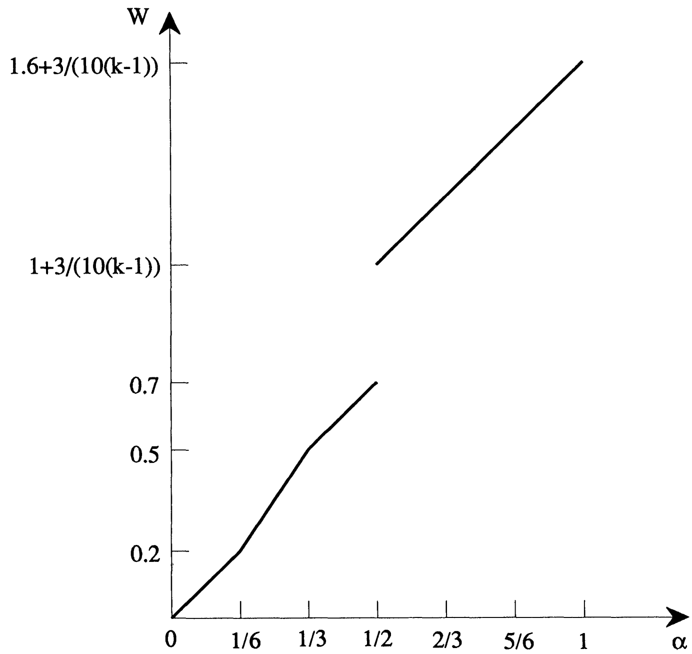


FIG. 2. *The weighting function $W(\alpha)$.*

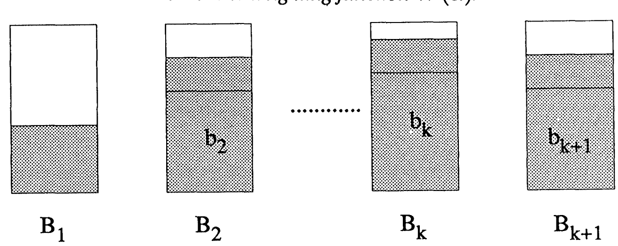


FIG. 3. *The possible packing when $B_1 \le \frac{1}{2}$.*

If the other item in $A_{2,1}$ is also in $(\frac{1}{6}, \frac{1}{3}]$, then

$$\begin{aligned}
&\tfrac{6}{5}B_1 + W(B_2) \\
&\ge \tfrac{6}{5}B_1 + \tfrac{6}{5}B_2 + \tfrac{3}{5}(1 - B_1) - \tfrac{1}{10} + \tfrac{3}{5}(1 - B_1) - \tfrac{1}{10} \\
&\ge 1 + \tfrac{6}{5}B_2.
\end{aligned}$$

If the other item is in $(\frac{1}{3}, \frac{1}{2}]$, then

$$\begin{aligned}
&\tfrac{6}{5}B_1 + W(B_2) \\
&\ge \tfrac{6}{5}B_1 + \tfrac{6}{5}B_2 + \tfrac{3}{5}(1 - B_1) - \tfrac{1}{10} + \tfrac{1}{10} \\
&\ge \tfrac{3}{5}B_1 + \tfrac{3}{5} + \tfrac{6}{5}B_2 \\
&\ge \tfrac{3}{5} \times \tfrac{2}{3} + \tfrac{3}{5} + \tfrac{6}{5}B_2 \\
&\ge 1 + \tfrac{6}{5}B_2.
\end{aligned}$$

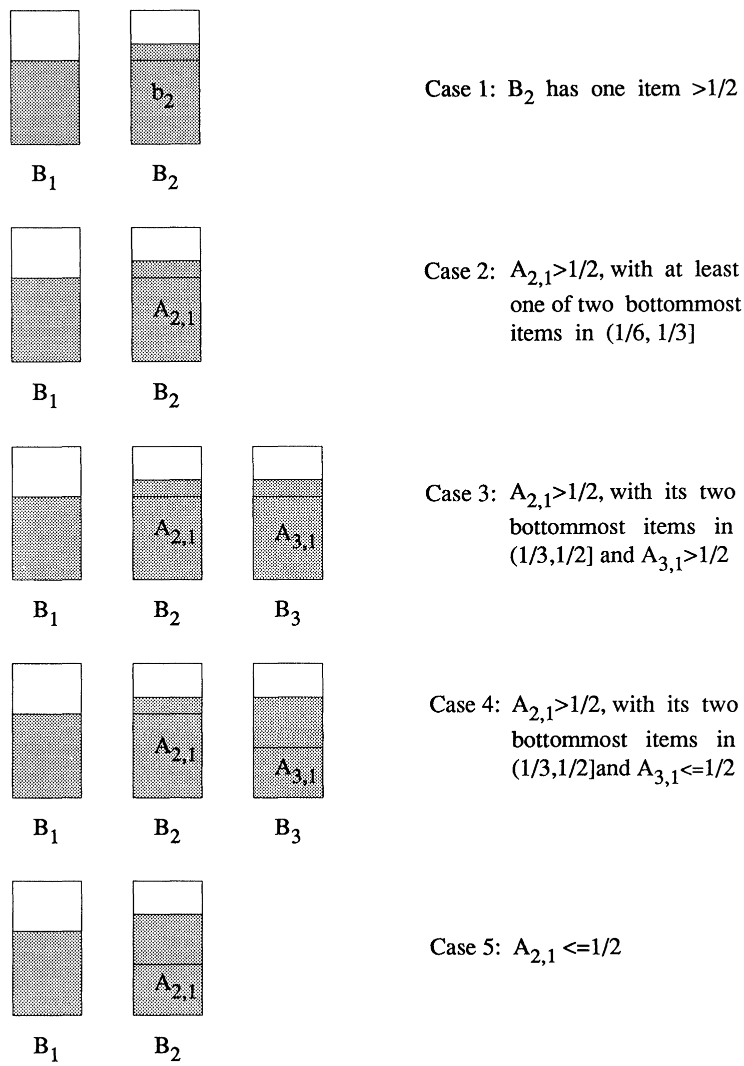


FIG. 4. *The possible packings when* $\frac{1}{2} < B_1 < \frac{5}{6}$.

*Case* 3. $A_{2,1} > \frac{1}{2}$, with its two bottommost items in $(\frac{1}{3}, \frac{1}{2}]$, and $A_{3,1} > \frac{1}{2}$. It is clear that $B_2 > \frac{2}{3}$.

If $A_{3,1}$ has one item greater than $\frac{1}{2}$, then

$$\begin{aligned}
&\tfrac{6}{5}B_1 + W(B_2) + W(B_3) \\
&\geq \tfrac{6}{5}B_1 + \tfrac{6}{5}B_2 + \tfrac{1}{10} + \tfrac{1}{10} + \tfrac{6}{5}B_3 + \tfrac{2}{5} + \tfrac{3}{10(k-1)} \\
&\geq \tfrac{6}{5} \times \tfrac{1}{2} + \tfrac{6}{5} \times \tfrac{2}{3} + \tfrac{3}{5} + \tfrac{6}{5}B_3 \\
&\geq 2 + \tfrac{6}{5}B_3.
\end{aligned}$$

If the two bottommost items of $A_{3,1}$ are in $(\frac{1}{6}, \frac{1}{3}]$, then

$$\begin{aligned}
&\tfrac{6}{5}B_1 + W(B_2) + W(B_3) \\
&\geq \tfrac{6}{5}B_1 + \tfrac{6}{5}B_2 + \tfrac{1}{10} + \tfrac{1}{10} + \tfrac{6}{5}B_3 + \tfrac{3}{5}(1 - B_1) - \tfrac{1}{10} + \tfrac{3}{5}(1 - B_2) - \tfrac{1}{10} \\
&\geq \tfrac{3}{5}B_1 + \tfrac{3}{5}B_2 + \tfrac{6}{5} + \tfrac{6}{5}B_3 \\
&\geq \tfrac{3}{5}B_1 + \tfrac{3}{5}(1 - B_1 + \tfrac{1}{3}) + \tfrac{6}{5} + \tfrac{6}{5}B_3 \\
&\geq 2 + \tfrac{6}{5}B_3.
\end{aligned}$$

If one of the two bottommost items is in $(\frac{1}{6}, \frac{1}{3}]$, and the other is in $(\frac{1}{3}, \frac{1}{2}]$, then

$$\begin{aligned}
&\tfrac{6}{5}B_1 + W(B_2) + W(B_3) \\
&\geq \tfrac{6}{5}B_1 + \tfrac{6}{5}B_2 + \tfrac{1}{10} + \tfrac{1}{10} + \tfrac{6}{5}B_3 + \tfrac{3}{5}(1 - B_2) - \tfrac{1}{10} + \tfrac{1}{10} \\
&\geq \tfrac{6}{5}B_1 + \tfrac{3}{5}B_2 + \tfrac{4}{5} + \tfrac{6}{5}B_3 \\
&\geq \tfrac{6}{5}B_1 + \tfrac{3}{5}(1 - B_1 + 1 - B_1) + \tfrac{4}{5} + \tfrac{6}{5}B_3 \\
&\geq 2 + \tfrac{6}{5}B_3.
\end{aligned}$$

If the two bottommost items are in $(\frac{1}{3}, \frac{1}{2}]$, then

$$\begin{aligned}
&\tfrac{6}{5}B_1 + W(B_2) + W(B_3) \\
&\geq \tfrac{6}{5}B_1 + \tfrac{6}{5}B_2 + \tfrac{1}{10} + \tfrac{1}{10} + \tfrac{6}{5}B_3 + \tfrac{1}{10} + \tfrac{1}{10} \\
&\geq \tfrac{6}{5}B_1 + \tfrac{6}{5}(1 - B_1 + \tfrac{1}{3}) + \tfrac{2}{5} + \tfrac{6}{5}B_3 \\
&\geq 2 + \tfrac{6}{5}B_3.
\end{aligned}$$

*Case* 4. $A_{2,1} > \frac{1}{2}$, with its two bottommost items in $(\frac{1}{3}, \frac{1}{2}]$, and $A_{3,1} \leq \frac{1}{2}$. In this case, we need to consider several possibilities according to the area distribution of $B_3$. In Fig. 5, on the right side of the vertical line are the three such possible packings that may follow the bins $B_1$ and $B_2$.

If $A_{3,1} + \cdots + A_{3,h} \leq \frac{1}{2}$, but $A_{3,1} + \cdots + A_{3,h+1} > \frac{1}{2}$, for $1 \leq h \leq k - 2$, then

$$\begin{aligned}
&\tfrac{6}{5}B_1 + W(B_2) + \cdots + W(B_{h+3}) \\
&\geq \tfrac{6}{5}B_1 + \tfrac{6}{5}B_2 + \tfrac{1}{10} + \tfrac{1}{10} + \tfrac{6}{5}B_3 + \tfrac{6}{5}(b_4 + \cdots + b_{h+2}) + (\tfrac{2}{5} + \tfrac{3}{10(k-1)})h + \tfrac{6}{5}B_{h+3} \\
&\geq \tfrac{6}{5}B_1 + \tfrac{6}{5}B_2 + \tfrac{1}{5} + \tfrac{6}{5}(A_{3,1} + A_{3,h+1}) + \tfrac{6}{5} \times \tfrac{1}{2} \times (h - 1) + \tfrac{2}{5}h + \tfrac{6}{5}B_{h+3} \\
&\geq \tfrac{6}{5}B_1 + \tfrac{6}{5}B_2 + \tfrac{6}{5}(1 - B_1 + 1 - B_2) + h - \tfrac{2}{5} + \tfrac{6}{5}B_{h+3} \\
&\geq h + 2 + \tfrac{6}{5}B_{h+3}.
\end{aligned}$$

If $A_{3,1} + \cdots + A_{3,k-1} \leq \frac{1}{2}$, but $A_{3,1} + \cdots + A_{3,k} > \frac{1}{2}$, then

$$\begin{aligned}
&\tfrac{6}{5}B_1 + W(B_2) + \cdots + W(B_{k+2}) \\
&\geq \tfrac{6}{5}B_1 + \tfrac{6}{5}B_2 + \tfrac{2}{10} + \tfrac{6}{5}B_3 + \tfrac{6}{5}(b_4 + \cdots + b_{k+1}) + (\tfrac{2}{5} + \tfrac{3}{10(k-1)})(k - 1) + \tfrac{6}{5}B_{k+2} \\
&\geq \tfrac{6}{5}B_1 + \tfrac{6}{5}(1 - B_1 + \tfrac{1}{3}) + \tfrac{1}{5} + \tfrac{6}{5} \times \tfrac{1}{2} + \tfrac{6}{5} \times \tfrac{1}{2} \times (k - 2) + \tfrac{2}{5}(k - 1) + \tfrac{3}{10} + \tfrac{6}{5}B_{k+2} \\
&\geq k + 1 + \tfrac{6}{5}B_{k+2}.
\end{aligned}$$

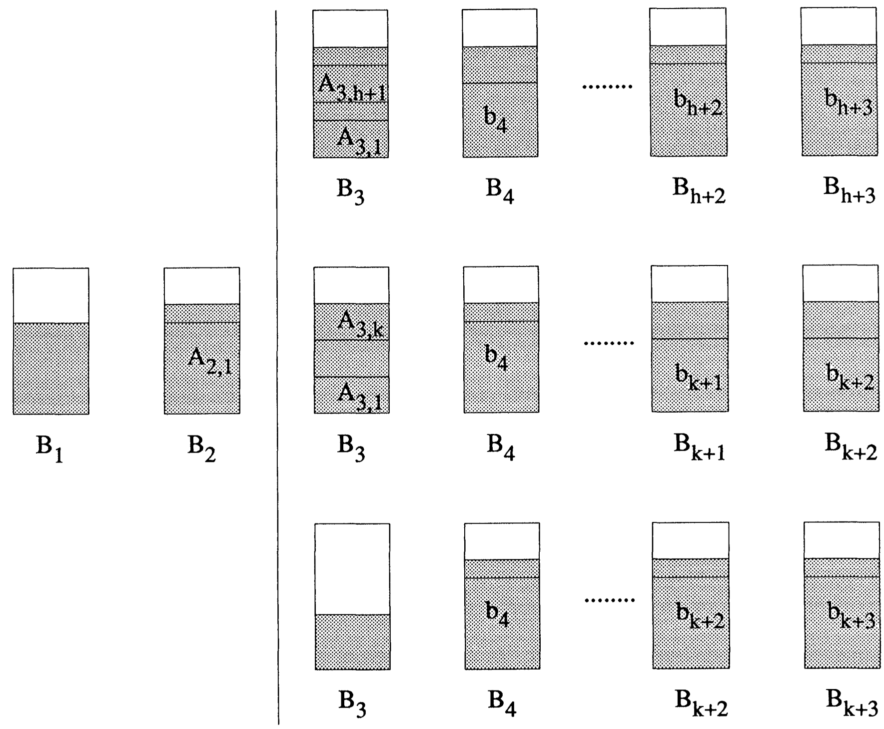


FIG. 5. *The possible packings when* $A_{2,1} > \frac{1}{2}$, *with its two bottommost items in* $(\frac{1}{3}, \frac{1}{2}]$, *and* $A_{3,1} \le \frac{1}{2}$.

If $A_{3,1} + \cdots + A_{3,k} \le \frac{1}{2}$, then

$$\begin{aligned}
&\tfrac{6}{5}B_1 + W(B_2) + \cdots + W(B_{k+3}) \\
&\ge \tfrac{6}{5}B_1 + \tfrac{6}{5}B_2 + \tfrac{1}{10} + \tfrac{1}{10} + \tfrac{6}{5}B_3 + \tfrac{6}{5}(b_4 + \cdots + b_{k+2}) + \left(\tfrac{2}{5} + \tfrac{3}{10(k-1)}\right)k + \tfrac{6}{5}B_{k+3} \\
&\ge \tfrac{6}{5}B_1 + \tfrac{6}{5}\left(1 - B_1 + \tfrac{1}{3}\right) + \tfrac{1}{5} + \tfrac{6}{5}A_{3,1} + \tfrac{6}{5}b_4 + \tfrac{6}{5} \times \tfrac{1}{2} \times (k-2) + \tfrac{2}{5}k + \tfrac{3}{10} + \tfrac{6}{5}B_{k+3} \\
&\ge \tfrac{6}{5}(A_{3,1} + b_4) + k + \tfrac{9}{10} + \tfrac{6}{5}B_{k+3} \\
&\ge \tfrac{6}{5} \times 1 + k + \tfrac{9}{10} + \tfrac{6}{5}B_{k+3} \\
&\ge k + 2 + \tfrac{6}{5}B_{k+3}.
\end{aligned}$$

*Case* 5. $A_{2,1} \le \frac{1}{2}$. Let us consider the subcases in Fig. 6.

If $A_{2,1} + \cdots + A_{2,h} \le \frac{1}{2}$, but $A_{2,1} + \cdots + A_{2,h+1} > \frac{1}{2}$, where $1 \le h \le k-2$, then it is easy to prove that $W(B_2) \ge \frac{6}{5}a + \frac{2}{5}$, where $a$ is the smallest item among $A_{2,1}, \ldots, A_{2,h+1}$. Because we know that $A_{2,1}$ and $A_{2,h+1}$ are both nonzero, so there are at least two items in these areas. Since $A_{2,1} + \cdots + A_{2,h+1} > \frac{1}{2}$, then $(A_{2,1} + \cdots + A_{2,h+1}) - a > \frac{1}{4}$. If there is one item in $A_{2,1}, \ldots, A_{2,h+1}$ in $(\frac{1}{3}, \frac{1}{2}]$, then $W(B_2) \ge \frac{6}{5}a + \frac{6}{5}((A_{2,1} + \cdots + A_{2,h+1}) - a) + \frac{1}{10} > \frac{6}{5}a + \frac{2}{5}$. Otherwise, all numbers in $A_{2,1}, \ldots, A_{2,h+1}$ are in $(\frac{1}{6}, \frac{1}{3}]$, and there are at least two of them. If there are only two numbers in $(\frac{1}{6}, \frac{1}{3}]$, then $W(B_2) \ge \frac{6}{5}a + \frac{6}{5}((A_{2,1} + \cdots + A_{2,h+1}) - a) + \frac{3}{5}(A_{2,1} + \cdots + A_{2,h+1}) - \frac{2}{10} > \frac{6}{5}a + \frac{6}{5} \times \frac{1}{4} + \frac{3}{5} \times \frac{1}{2} - \frac{2}{10} = \frac{6}{5}a + \frac{2}{5}$. If $A_{2,1}, \ldots, A_{2,h+1}$ have at least three items in $(\frac{1}{6}, \frac{1}{3}]$, then $W(B_2) \ge \frac{6}{5}a + \frac{6}{5} \times (\frac{1}{6} + \frac{1}{6}) = \frac{6}{5}a + \frac{2}{5}$. Therefore,

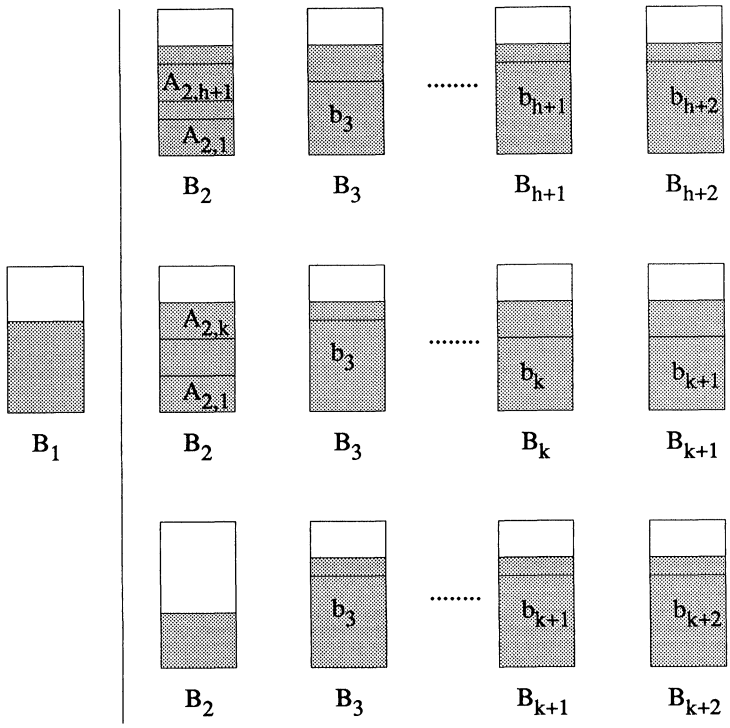


FIG. 6. *The possible packings when* $A_{2,1} \le \frac{1}{2}$.

$$
\begin{aligned}
&\tfrac{6}{5}B_1 + W(B_2) + \cdots + W(B_{h+2}) \\
&\quad \ge \tfrac{6}{5}B_1 + \tfrac{6}{5}a + \tfrac{2}{5} + \tfrac{6}{5}(b_3 + \cdots + b_{h+1}) + (\tfrac{2}{5} + \tfrac{3}{10(k-1)})h + \tfrac{6}{5}B_{h+2} \\
&\quad \ge \tfrac{6}{5}(B_1 + a) + \tfrac{2}{5} + \tfrac{6}{5} \times \tfrac{1}{2} \times (h-1) + \tfrac{2}{5}h + \tfrac{6}{5}B_{h+2} \\
&\quad \ge \tfrac{6}{5} \times 1 + \tfrac{2}{5} + \tfrac{3}{5}(h-1) + \tfrac{2}{5}h + \tfrac{6}{5}B_{h+2} \\
&\quad \ge h + 1 + \tfrac{6}{5}B_{h+2}.
\end{aligned}
$$

If $A_{2,1} + \cdots + A_{2,k-1} \le \frac{1}{2}$, but $A_{2,1} + \cdots + A_{2,k} > \frac{1}{2}$, then it is easy to prove that $W(B_2) \ge \frac{6}{5}A_{2,1} + \frac{1}{10}$. Because if $A_{2,1}$ has at least one item in $(\frac{1}{3}, \frac{1}{2}]$, then the inequality is obvious. If all the items in $A_{2,1}$ are in $(\frac{1}{6}, \frac{1}{3}]$, then there are at most two such items in $A_{2,1}$ since $A_{2,1}$ is less than $\frac{1}{2}$. So $W(B_2) \ge \frac{9}{5}A_{2,1} - \frac{1}{10} \times 2 + \frac{6}{5}(A_{2,2} + \cdots + A_{2,k}) > \frac{6}{5}A_{2,1} - \frac{1}{5} + \frac{3}{5}B_2 > \frac{6}{5}A_{2,1} - \frac{1}{5} + \frac{3}{5} \times \frac{1}{2} = \frac{6}{5}A_{2,1} + \frac{1}{10}$. Therefore,

$$
\begin{aligned}
&\tfrac{6}{5}B_1 + W(B_2) + \cdots + W(B_{k+1}) \\
&\quad \ge \tfrac{6}{5}B_1 + \tfrac{6}{5}A_{2,1} + \tfrac{1}{10} + \tfrac{6}{5}(b_3 + \cdots + b_k) + (\tfrac{2}{5} + \tfrac{3}{10(k-1)})(k-1) + \tfrac{6}{5}B_{k+1} \\
&\quad \ge \tfrac{6}{5} \times \tfrac{1}{2} + \tfrac{6}{5}(A_{2,1} + b_3) + \tfrac{1}{10} + \tfrac{6}{5} \times \tfrac{1}{2} \times (k-3) + \tfrac{2}{5}(k-1) + \tfrac{3}{10} + \tfrac{6}{5}B_{k+1} \\
&\quad \ge \tfrac{3}{5} + \tfrac{6}{5} \times 1 + \tfrac{1}{10} + \tfrac{3}{5}(k-3) + \tfrac{2}{5}(k-1) + \tfrac{3}{10} + \tfrac{6}{5}B_{k+1} \\
&\quad \ge k + \tfrac{6}{5}B_{k+1}.
\end{aligned}
$$

If $A_{2,1} + \cdots + A_{2,k} \le \frac{1}{2}$, then it is easy to prove that $W(B_2) \ge \frac{6}{5}B_2 + \frac{3}{5}A_{2,1} - \frac{1}{5}$. Because if $A_{2,1}$ has at least one item in $(\frac{1}{3}, \frac{1}{2}]$, then $W(B_2) \ge \frac{6}{5}B_2 + \frac{1}{10} = \frac{6}{5}B_2 + \frac{3}{5} \times \frac{1}{2} - \frac{1}{5} \ge \frac{6}{5}B_2 + \frac{3}{5}A_{2,1} - \frac{1}{5}$. If all the numbers in $A_{2,1}$ are in $(\frac{1}{6}, \frac{1}{3}]$, then $W(B_2) \ge \frac{9}{5}A_{2,1} - \frac{1}{10} \times 2 + \frac{6}{5}(A_{2,2} + \cdots + A_{2,k}) = \frac{6}{5}B_2 + \frac{3}{5}A_{2,1} - \frac{1}{5}$. Therefore,

$$\begin{aligned}
&\tfrac{6}{5}B_1 + W(B_2) + \cdots + W(B_{k+2}) \\
&\ge \tfrac{6}{5}B_1 + \tfrac{6}{5}B_2 + \tfrac{3}{5}A_{2,1} - \tfrac{1}{5} + \tfrac{6}{5}(b_3 + \cdots + b_{k+1}) + (\tfrac{2}{5} + \tfrac{3}{10(k-1)})k + \tfrac{6}{5}B_{k+2} \\
&\ge \tfrac{3}{5}B_1 + \tfrac{3}{5}(B_1 + A_{2,1}) - \tfrac{1}{5} + \tfrac{6}{5}(B_2 + b_3) + \tfrac{6}{5} \times \tfrac{1}{2} \times (k-2) + \tfrac{2}{5}k + \tfrac{3}{10} + \tfrac{6}{5}B_{k+2} \\
&\ge \tfrac{3}{5} \times \tfrac{1}{2} + \tfrac{3}{5} \times 1 - \tfrac{1}{5} + \tfrac{6}{5} \times 1 + \tfrac{3}{5}(k-2) + \tfrac{2}{5}k + \tfrac{3}{10} + \tfrac{6}{5}B_{k+2} \\
&\ge k + 1 + \tfrac{6}{5}B_{k+2}.
\end{aligned}$$

This ends the case analysis. If beginning with $B_j$ ($B_1$ in the case analysis) there is a portion of the $NkF$ packing which matches one of the above cases, and if we let $l$ be the index of the last bin in that portion minus $j$, then $j + l \le NkF(L)$, and $\frac{6}{5}B_j + W(B_{j+1}) + \cdots + W(B_{j+l}) \ge l + \frac{6}{5}B_{j+l}$, which satisfies (1) in the Lemma. However, if the $NkF$ packing of the list $L$ ends without completely matching any of the above cases, i.e., $B_j, \ldots, B_{NkF(L)}$ only matches the first part of one of the cases, then we can see that no matter where the packing ends $B_j$ is followed by $h(\ge 0)$ bins with $\frac{6}{5}B_j + W(B_{j+1}) + \cdots + W(B_{j+h}) \ge h$, then followed by $g(\ge 0)$ bins with items greater than $\frac{1}{2}$, hence each having weight greater than 1. If we let $l$ be the index of the last bin in the packing minus $j$, i.e., $NkF(L) - j$, then $j + l = NkF(L)$, and $\frac{6}{5}B_j + W(B_{j+1}) + \cdots + W(B_{NkF(L)}) + 2 \ge (NkF(L) - j) + 2 \ge l + \frac{6}{5}B_{NkF(L)}$, which satisfies (2) in the Lemma.

**4. Proof of the main theorem.**

CLAIM 1. *For any bin $B_i$ of items of total size 1 or less*,

$$W(B_i) \le 1.7 + \frac{3}{10(k-1)}.$$

*Proof.* See the proof of Lemma 1 in the work of Garey, Graham, Johnson, and Yao [4]. We note that our weighting function differs from that in the reference only by the addition of $\frac{3}{10(k-1)}$ for the items of size exceeding $\frac{1}{2}$, and there can be only one such item in $B_i$. So the bound in the claim exceeds the bound 1.7 in the reference by precisely this amount. □

CLAIM 2. *For any list $L$*,

$$W(L) \le \left(1.7 + \frac{3}{10(k-1)}\right) L^*.$$

*Proof.* Apply the optimal algorithm to $L$. We get $L^*$ nonempty bins.

$$\begin{aligned}
W(L) &= \textstyle\sum_{i=1}^{L^*} W(B_i) \\
&\le \textstyle\sum_{i=1}^{L^*} (1.7 + \frac{3}{10(k-1)}) \qquad \text{(by Claim 1)} \\
&= (1.7 + \tfrac{3}{10(k-1)})L^*.
\end{aligned}$$

□

CLAIM 3. *For any list $L$, there exists a constant $c$ such that*

$$W(L) + c \ge NkF(L).$$

*Proof.* Let $j$ be the largest index of the bins in the $NkF$ packing such that $\sum_{i=1}^{j} W(B_i) \geq j - 1 + \frac{6}{5}B_j$. Such $j$ always exists.

If $j = NkF(L)$, then $W(L) = \sum_{i=1}^{j} W(B_i) \geq j - 1 + \frac{6}{5}B_j \geq NkF(L) - 1$. So $W(L) + 1 \geq NkF(L)$. Now assume $j < NkF(L)$. Let us consider $B_j$.

If $B_j \geq \frac{5}{6}$, then $\sum_{i=1}^{j} W(B_i) + W(B_{j+1}) \geq j - 1 + \frac{6}{5}B_j + \frac{6}{5}B_{j+1} \geq j + \frac{6}{5}B_{j+1}$. There exists $j+1$, such that $\sum_{i=1}^{j+1} W(B_i) \geq j + \frac{6}{5}B_{j+1}$. This is a contradiction to the assumption that $j$ is the largest index having the property. So the case of $B_j \geq \frac{5}{6}$ can never happen.

If $B_j < \frac{5}{6}$, and (1) in Lemma happens, then $\sum_{i=1}^{j} W(B_i) + W(B_{j+1}) + \cdots + W(B_{j+l}) \geq j-1+\frac{6}{5}B_j+l+\frac{6}{5}B_{j+l}-\frac{6}{5}B_j$. Therefore, $\sum_{i=1}^{j+l} W(B_i) \geq j+l-1+\frac{6}{5}B_{j+l}$. This is again a contradiction to the assumption that $j$ is the largest index. So (1) in Lemma can never happen.

If $B_j < \frac{5}{6}$, and (2) in Lemma happens, then we have $\sum_{i=1}^{j} W(B_i) + W(B_{j+1}) + \cdots + W(B_{NkF(L)}) + 2 \geq j - 1 + \frac{6}{5}B_j + NkF(L) - j + \frac{6}{5}B_{NkF(L)} - \frac{6}{5}B_j$. So $W(L) + 3 \geq NkF(L)$. $\square$

Now we are prepared to prove Main Theorem.

*Proof of Main Theorem.*

$$\begin{aligned} R[NkF] &= \limsup \max\{NkF(L)/L^*\} \\ &\leq \lim_{L^* \to \infty} (W(L) + c)/L^* \quad \text{(by Claim 3)} \\ &\leq \lim_{L^* \to \infty} ((1.7 + \tfrac{3}{10(k-1)})L^* + c)/L^* \quad \text{(by Claim 2)} \\ &= 1.7 + \tfrac{3}{10(k-1)}. \end{aligned}$$

Combining with the previous results $R[N2F] = 2$ and $R[NkF] \geq 1.7 + \frac{3}{10(k-1)}$, we have $R[NkF] = 1.7 + \frac{3}{10(k-1)}$ for $k \geq 2$.

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